

Addendum P201

PSLQ Hunt on the Four-Loop Coefficient c_4

P18-T2 Programme: G_2 -structured Perturbative Expansion for m_μ/m_e

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1 Context

The P18-T2 programme constructs the muon-to-electron mass ratio from first TOE principles. Paper P199 established a G_2 -structured three-loop formula R_3 for m_μ/m_e using the exact TOE constants

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \quad \Omega_0 = \frac{\pi^3}{4}, \quad (1)$$

arriving at a relative residual $\Delta_{\text{rel}} \approx 8.81 \times 10^{-7}$. Paper P200 confirmed that no simple G_2 -motivated closed form closes that gap exactly over three loops, and extracted the four-loop coefficient

$$c_4 = \frac{m_\mu/m_e - R_3}{207 \alpha^4}$$

as the object to be characterised. The present addendum carries out an exhaustive PSLQ integer-relation search over two prescribed bases $\mathcal{B}_1, \mathcal{B}_2$ and a targeted rational scan, reports all near-coincidences, and concludes with the open-problem declaration **OP-C4**.

All computations use `mpmath` at `dps` = 60 (approximately 60 significant decimal digits). The CODATA 2018 central value $m_\mu/m_e = 206.7682830$ is taken as the target; its quoted uncertainty $\pm 4.6 \times 10^{-6}$ propagates to an uncertainty in c_4 of ± 7.84 , so c_4 is experimentally constrained to only ~ 4.7 significant figures.

2 The Four-Loop Coefficient c_4

2.1 Setup

The three-loop G_2 formula from P199 is

$$R_3 = 207 \left(1 - \frac{\alpha}{2\pi} + f_2 \alpha^2 + N \alpha^3 \right), \quad (2)$$

where

$$f_2 = \frac{1}{\pi^2} + \frac{11}{16}, \quad (3)$$

$$N = -\frac{11}{4} \Omega_0 \alpha^{-1} = -\frac{11\pi^3}{16} \alpha^{-1} \approx -2921.1776. \quad (4)$$

Evaluating at $\alpha^{-1} = 137.036303775878 \dots$:

$$R_3 = 206.533309024079 \dots, \quad (5)$$

$$\Delta_{\text{abs}} \equiv m_\mu/m_e - R_3 = 0.234973975920290 \dots, \quad (6)$$

$$\Delta_{\text{rel}} = \Delta_{\text{abs}}/(m_\mu/m_e) \approx 1.136 \times 10^{-3}. \quad (7)$$

The four-loop coefficient is

$$c_4 = \frac{\Delta_{\text{abs}}}{207 \alpha^4} = \frac{m_\mu/m_e - R_3}{207 \alpha^4}. \quad (8)$$

2.2 Numerical value to 30 significant figures

$$\boxed{c_4 = 400305.875316291313324816036107 \dots} \quad (9)$$

The integer part is 400305 and the fractional part is 0.875316291313.... The CODATA uncertainty propagates as

$$\delta c_4 = \frac{\delta(m_\mu/m_e)}{207 \alpha^4} \approx \pm 7.84,$$

so only the leading digits $c_4 \approx 4.003 \times 10^5$ are experimentally well-constrained. Any closed-form candidate must agree with this value to significantly better than ± 7.84 to be considered non-accidental.

2.3 Sensitivity

The conversion factor $1/(207 \alpha^4) \approx 1.703 \times 10^6$ means that a 1×10^{-7} change in the input m_μ/m_e shifts c_4 by ± 0.17 . This high sensitivity is an intrinsic feature of extracting four-loop coefficients from two-part-per-million experimental data.

3 PSLQ Search over Basis $\mathcal{B}_1$

3.1 Basis definition

The 13-element basis $\mathcal{B}_1$ (with linearly dependent pure integers removed to avoid degenerate relations) consists of:

$$\mathcal{B}_1 = \{c_4, 1, \pi, \pi^2, \alpha^{-1}, \Omega_0, \alpha^{-1}\Omega_0, \alpha^{-2}, \Omega_0^2, \ln 2, \ln \pi, \gamma_E, \zeta(3)\}, \quad (10)$$

where γ_E is the Euler–Mascheroni constant and $\zeta(3) = 1.20206\dots$ is Apéry’s constant.

Note on independence. Because $\Omega_0 = \pi^3/4$ the powers π^3 and π^4 are linearly dependent on $\{\pi, \alpha^{-1}, \Omega_0\}$ over $\mathbb{Q}$, so including them explicitly would allow PSLQ to find trivial internal relations before ever reaching c_4 . We therefore omit π^3 and π^4 from $\mathcal{B}_1$ and keep only algebraically independent combinations.

3.2 Result

`mpmath.pslq( $\mathcal{B}_1$ , maxcoeff=2000, tol= $10^{-20}$ )` returns `None` — no integer relation found.

Proposition 3.1. $c_4 \notin \mathbb{Z}\text{-span}\{\mathcal{B}_1 \setminus \{c_4\}\}$ with all integer coefficients satisfying $|n_i| \leq 2000$.

4 PSLQ Search over Basis $\mathcal{B}_2$

4.1 Basis definition

The 13-element products basis is:

$$\mathcal{B}_2 = \left\{c_4, 1, \pi, \frac{\alpha^{-1}}{\pi}, \frac{\Omega_0}{\pi}, \frac{\alpha^{-1}\Omega_0}{\pi^2}, \frac{\alpha^{-2}}{\pi^2}, (\alpha^{-1}\Omega_0)^2, \sqrt{\alpha^{-1}}, \alpha^{-3/2}, 207\Omega_0, 207\pi, \frac{207}{\alpha^{-1}}\right\}. \quad (11)$$

4.2 Result

`mpmath.pslq( $\mathcal{B}_2$ , maxcoeff=1000, tol= $10^{-20}$ )` finds a relation with c_4 -coefficient zero, namely the identity $\alpha^{-1}/\pi - 16\Omega_0/\pi - \pi - 1 = 0$ (which follows from $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ and $\Omega_0 = \pi^3/4$). No relation involving c_4 is found.

Proposition 4.1. $c_4 \notin \mathbb{Z}\text{-span}\{\mathcal{B}_2 \setminus \{c_4\}\}$ with all integer coefficients satisfying $|n_i| \leq 1000$.

5 Rational Approximation Scan

5.1 Near-integer test

$c_4 \approx 400305.875\dots$ is not near an integer; the nearest integer is 400306 with a gap of 0.1247.

5.2 Small-denominator scan

Scanning all p/q with $q \leq 32$ reveals:

Candidate p/q	Value	$ c_4 - p/q $	Rel. error
400306/1	400306	0.1247	3.1×10^{-7}
2001529/5	400305.8	0.0753	1.9×10^{-7}
6004588/15	400305.867	0.00865	2.2×10^{-8}
3202447/8	400305.875	3.16×10^{-4}	7.9×10^{-10}

The best rational match with $q \leq 32$ is

$$c_4 \approx \frac{3202447}{8} = 400305.875, \quad |c_4 - 3202447/8| = 3.16 \times 10^{-4}.$$

This gap is 4×10^{-5} of the CODATA uncertainty, making the match consistent with the experimental precision. However, no theoretical derivation of the denominator 8 from the TOE structure is known.

5.3 TOE ratio tests

Ratio	Value	Dist. from nearest rational ($ n \leq 10$)
c_4/α^{-1}	2921.167	0.167 from 2921
c_4/α^{-2}	21.317	0.317 from 21
c_4/Ω_0	51641.9	0.085 from 51642
$c_4/(\alpha^{-1}\Omega_0)$	376.85	0.152 from 377

None of these ratios is within 10^{-4} of a rational with small denominator.

5.4 Near-coincidence with $-N\alpha^{-1}$

The most structurally significant observation is:

$$c_4 \approx -N\alpha^{-1} = \frac{11}{4}\Omega_0\alpha^{-2} = \frac{11\pi^3}{16}\alpha^{-2}, \quad (12)$$

where $-N\alpha^{-1} = 400307.376\dots$ and the residual is

$$c_4 - (-N\alpha^{-1}) = -1.5010\dots \approx -0.19\delta c_4. \quad (13)$$

This residual of 0.19 CODATA standard deviations is formally consistent with measurement uncertainty but not zero. The near-coincidence suggests that the four-loop structure may couple to the same G_2 weight 11 and Ω_0 factors that appear at three loops, but this remains an observation rather than a theorem.

6 Verdict and Open Problem

6.1 Summary of search

Search	Basis	Max coeff	Result
<code>mpmath.identify(c_4)</code>	—	—	No meaningful hit
PSLQ over $\mathcal{B}_1$	13 elements	2000	None
PSLQ over $\mathcal{B}_2$	13 elements	1000	None (trivial only)
Rational scan p/q , $q \leq 1000$	—	—	Best err 1.4×10^{-6}
Pi-polynomial basis $\{1, \pi, \dots, \pi^{12}\}$	13 elements	1000	None
Targeted (N -products) basis	10 elements	1000	None (trivial only)

No PSLQ relation involving c_4 was found over any of the bases searched. The combined bound is:

$$c_4 \notin \mathbb{Z}\text{-span}\{\mathcal{B}_1\} \text{ with } |n_i| \leq 2000.$$

6.2 Numerical anchor

For future searches at higher loop order or with improved experimental input, the anchor value is

$$c_4 = 400305.875316291313324816036107\dots \quad (14)$$

with experimental uncertainty ± 7.84 from CODATA 2018.

6.3 Open Problem

Open Problem 6.1 (OP-C4). Does c_4 admit a closed-form expression in terms of the TOE constants $\{\alpha^{-1}, \Omega_0, \pi, \ln 2, \gamma_E, \zeta(3)\}$ with integer or small-rational coefficients? More precisely: does there exist a finite set of integers n_i and algebraic combinations b_i of the above constants such that

$$c_4 = \sum_i n_i b_i,$$

with the identity holding to precision exceeding 10^{-15} ?

Sub-questions.

- (i) Is the near-coincidence $c_4 \approx (11/4)\Omega_0\alpha^{-2}$ (residual ≈ -1.50 , i.e. $0.19\delta c_4$) exact in the TOE amplitude, with the discrepancy attributable to higher-order corrections or an as-yet-unidentified loop factor?
- (ii) Is the rational near-match $c_4 \approx 3202447/8$ (residual 3.16×10^{-4} , well within CODATA uncertainty) derivable from the TOE integer spectrum?
- (iii) Does the coefficient c_4 receive contributions from new Lie-algebraic structures beyond G_2 (e.g. F_4 or E_6) that do not appear in the bases $\mathcal{B}_1, \mathcal{B}_2$?

Resolution of OP-C4 requires either a PSLQ hit at higher coefficient bound ($|n_i| > 2000$), an improved experimental measurement of m_μ/m_e , or a first-principles TOE derivation of the four-loop amplitude.

6.4 Four-loop formula record

For completeness, the four-loop formula with the numerical anchor value is:

$$R_4 = 207 \left(1 - \frac{\alpha}{2\pi} + f_2 \alpha^2 + N \alpha^3 + c_4 \alpha^4 \right), \quad (15)$$

with $f_2 = 1/\pi^2 + 11/16$, $N = -(11/4)\Omega_0\alpha^{-1}$, and $c_4 = 400305.875316\dots$. By construction $R_4 = m_\mu/m_e = 206.7682830$ identically (at the precision of the input), so R_4 does not reduce the experimental residual — it merely records the four-loop term that would be required by a perturbative expansion.

The remaining gap at four loops is:

$$\Delta_{\text{rel}}^{(4)} = |R_4 - m_\mu/m_e|/(m_\mu/m_e) = 0,$$

by construction. The physical content of c_4 is determined only once an *independent* theoretical prediction for it exists.