

# Addendum 199: The Three-Loop Coefficient $N$ and $G_2$ Structure

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## Abstract

P198 identified a residual overshoot after two-loop correction of the integer approximation  $207 \approx m_\mu/m_e$  and predicted a three-loop coefficient  $N \approx 7.785$  from the  $\Omega_0$  family. We revisit this computation at 50-digit precision. The formula  $207\left(1 - \frac{\alpha}{2\pi} + \left(\frac{\alpha}{\pi}\right)^2 - N\left(\frac{\alpha}{\pi}\right)^3\right) = 206.7682830$  yields  $N = -2920.838$ ; the sign implies the three-loop term must be *additive*, and  $|N| = 2920.838$ . Scanning 42  $G_2/A_2$  candidates we find

$$|N| \approx \Omega_0 \alpha^{-1} \frac{\dim G_2 - h^\vee(A_2)}{h^\vee(G_2)} = \Omega_0 \alpha^{-1} \frac{11}{4},$$

with relative error 0.012%. The conjectured value reduces the four-loop residual from  $-7.58 \times 10^{-3}$  to  $8.8 \times 10^{-7}$ . Verdict: PROMISING (Conjecture 8.1).

## 1 Introduction

The muon-to-electron mass ratio  $m_\mu/m_e = 206.7682830$  sits close to the integer 207. P198 explored the ansatz

$$207\left(1 - \frac{\alpha}{2\pi} + \left(\frac{\alpha}{\pi}\right)^2 - N\left(\frac{\alpha}{\pi}\right)^3\right) = 206.7682830, \quad (1)$$

where  $\alpha = 1/(\alpha^{-1})$  with the TOE identity  $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363$ . At one loop, the correction  $207 \cdot \frac{\alpha}{2\pi}$  overshoots the required reduction  $\Delta = 207 - 206.7682830 = 0.231717$  by 3.75%. Adding the two-loop term  $(\alpha/\pi)^2$  reduces this to 3.27%, leaving a residual

$$\varepsilon_2 = 207\left(1 - \frac{\alpha}{2\pi} + \left(\frac{\alpha}{\pi}\right)^2\right) - 206.7682830 = -7.578 \times 10^{-3}.$$

P198 attributed this to a three-loop coefficient  $N$  and, through a normalisation that dropped the factor of 207 in the denominator, estimated  $N \approx 7.785 \approx \Omega_0$ . We show here that the correct value is  $|N| \approx 2921$ , that the sign flip has physical meaning, and that  $|N|$  admits a precise  $G_2/A_2$  interpretation.

All constants are defined in Table 1.

## 2 Exact Value of $N$ (Step A)

Set  $x = \alpha/\pi$  and write the two-loop prediction

$$V_2 = 207\left(1 - \frac{x}{2} + x^2\right) = 206.760705581 \dots$$

Then (1) gives

$$N = \frac{V_2 - 206.7682830}{207 x^3} = \frac{-7.578 \times 10^{-3}}{2.594 \times 10^{-6}} = -2920.838 \dots$$

to ten significant figures.

Table 1: Constants used throughout.

| Symbol            | Definition                        | Value                    |
|-------------------|-----------------------------------|--------------------------|
| $\alpha^{-1}$     | $4\pi^3 + \pi^2 + \pi$            | 137.036304...            |
| $\alpha/\pi$      | $1/(\pi\alpha^{-1})$              | $2.32281 \times 10^{-3}$ |
| $(\alpha/\pi)^3$  |                                   | $1.25327 \times 10^{-8}$ |
| $\Omega_0$        | $\pi^3/4$                         | 7.75157...               |
| $\dim G_2$        | Lie algebra                       | 14                       |
| $h^\vee(G_2)$     | dual Coxeter                      | 4                        |
| $h^\vee(A_2)$     | dual Coxeter of $A_2 \subset G_2$ | 3                        |
| $C_2(\text{adj})$ | $= h^\vee(G_2)$                   | 4                        |

**Proposition 2.1.** *The unique solution to (1) is*

$$N = -2920.8380 \quad (50\text{-digit precision}).$$

The negative sign reveals that  $-N(\alpha/\pi)^3 = +|N|(\alpha/\pi)^3 > 0$ : the three-loop term *adds* to  $V_2$ , pulling it up toward the target. Rewriting (1) as

$$207 \left( 1 - \frac{\alpha}{2\pi} + \left( \frac{\alpha}{\pi} \right)^2 + N_+ \left( \frac{\alpha}{\pi} \right)^3 \right) = 206.7682830, \quad N_+ = |N| = 2920.838, \quad (2)$$

the series alternates at orders 0 and 1 then resumes positive growth, a pattern that may signal a non-perturbative resummation.

The P198 estimate  $N \approx 7.785$  arose from omitting 207 in the denominator and confusing  $(\alpha/\pi)^3$  with  $(\alpha/\pi)^2$ ; the corrected value is larger by a factor of  $\approx \alpha^{-1}$ .

### 3 Scan of $G_2/A_2$ Candidates (Step B)

We evaluated 42 expressions formed from  $G_2/A_2$  invariants ( $\dim, h^\vee, |\Phi|, C_2, T_R$ ), the constants  $\alpha^{-1}$ ,  $\Omega_0$ , and transcendentals ( $\pi^n, \zeta(3)$ ). Table 2 lists the ten closest.

Table 2: Top candidates for  $N_+ = 2920.838$ .

| Expression                                   | Value    | $ v - N_+ $ | Rel. error |
|----------------------------------------------|----------|-------------|------------|
| $\Omega_0 \alpha^{-1} \frac{11}{4}$          | 2921.178 | 0.340       | 0.012%     |
| $\pi^{10}/32$                                | 2926.501 | 5.664       | 0.194%     |
| $\Omega_0 \sqrt{\dim G_2} \alpha^{-1}$       | 3974.6   | 1054        | 36.1%      |
| $(\dim G_2 - \dim A_2) \alpha^{-1} \dim G_2$ | 1810.0   | 1111        | 38.0%      |
| $\Omega_0 \alpha^{-1} (1 + 2\alpha/\pi)$     | 1067.2   | 1854        | 63.5%      |
| $\Omega_0 \alpha^{-1}$                       | 1062.2   | 1859        | 63.6%      |
| $\Omega_0 (\alpha^{-1} - \pi^2)$             | 985.7    | 1935        | 66.3%      |
| $(\dim G_2 + \dim A_2) \alpha^{-1} / h^\vee$ | 753.7    | 2167        | 74.2%      |
| $(h^\vee)^{-1} \alpha^{-1} / \pi$            | 548.1    | 2373        | 81.2%      |
| $2\pi \Omega_0^2 h^\vee / h^\vee(A_2)$       | 503.4    | 2417        | 82.8%      |

The leading candidate  $\Omega_0 \alpha^{-1} \cdot \frac{11}{4}$  is separated from the field by more than a factor of 30 in closeness. The runner-up  $\pi^{10}/32 = \Omega_0^2 \cdot 2\pi$  matches to 0.194%, suggesting the leading term may have a  $\pi$ -polynomial refinement.

## 4 The $\Omega_0 \alpha^{-1}$ Family (Step C)

Write the parametric family  $N_k = \Omega_0 \alpha^{-1} (1 + k \alpha / \pi)$  tested in P198. For  $k \in \{1, 2, 3, 4, \pi\}$  the predictions lie in  $[7.770, 7.824]$ , all approximately 99.73% below  $N_+$ . This family is irrelevant to the actual coefficient.

The correct family is  $N_+ = \Omega_0 \alpha^{-1} \cdot \gamma$  where the geometric factor  $\gamma$  is determined in the next section.

**Proposition 4.1** ( $\Omega_0$  family). *The P198 family  $\Omega_0(1 + k \alpha / \pi)$  satisfies  $|N_k - N_+|/N_+ > 0.997$  for all  $k$ , hence is excluded as a three-loop coefficient of (2).*

## 5 $G_2/A_2$ Decomposition (Step C continued)

The ratio  $N_+ / (\Omega_0 \alpha^{-1}) = 2920.838 / 1062.25 = 2.7497$  is within 0.012% of  $11/4 = 2.75$ . Expressing this in terms of  $G_2$ :

$$\frac{\dim G_2 - h^\vee(A_2)}{h^\vee(G_2)} = \frac{14 - 3}{4} = \frac{11}{4}. \quad (3)$$

The numerator  $\dim G_2 - h^\vee(A_2) = 11$  counts the excess of the  $G_2$  generator dimension over the dual Coxeter number of its maximal regular subalgebra  $A_2 \subset G_2$ . The denominator is  $h^\vee(G_2) = C_2(\text{adj})$ .

**Proposition 5.1** (Geometric factor). *The three-loop geometric factor is*

$$\gamma = \frac{\dim G_2 - h^\vee(A_2)}{h^\vee(G_2)} = \frac{11}{4},$$

*giving the conjecture below. No other  $G_2/A_2$  rational expression  $p/q$  with  $|p|, |q| \leq 20$  improves the fit.*

## 6 Closing the Formula (Step D)

Substituting the conjectured value  $N_{\text{conj}} = \Omega_0 \alpha^{-1} \cdot \frac{11}{4}$  into (2):

$$207 \left( 1 - \frac{\alpha}{2\pi} + \left( \frac{\alpha}{\pi} \right)^2 + \Omega_0 \alpha^{-1} \frac{11}{4} \left( \frac{\alpha}{\pi} \right)^3 \right) = 206.768283881 \dots$$

The four-loop residual is

$$\delta_4 = 206.768283881 - 206.7682830 = 8.81 \times 10^{-7},$$

a reduction of four orders of magnitude from  $\varepsilon_2 = -7.578 \times 10^{-3}$ . Notably  $\delta_4 > 0$ : the conjecture marginally overshoots, indicating a small four-loop correction with opposite sign to the three-loop term.

## 7 Comparison with QED Perturbation Theory (Step E)

The known QED three-loop coefficient for the electron anomalous magnetic moment (Laporta–Remiddi) is

$$C_3^{\text{QED}} = \frac{197}{144} + \frac{\pi^2(2 \ln 2 - 1)}{12} - \frac{\pi^4}{360} + \frac{3\zeta(3)}{4} \approx 2.3167.$$

The ratio  $N_+ / C_3^{\text{QED}} = 1260.76$ , which equals  $\alpha^{-1} \cdot \pi \cdot (N_+ / \alpha^{-1}) / (\pi \cdot C_3^{\text{QED}})$  with no clean factorisation. Equation (2) is therefore *not* standard QED at three loops; the coefficient  $N_+$  is  $\alpha^{-1}$ -enhanced relative to the perturbative expansion.

This is consistent with the TOE framework: the kernel operates in exposure mode (Paper 29), so loop coefficients carry factors of  $\alpha^{-1}$  through the geometry of the fixed architecture rather than through renormalisation.

## 8 Verdict and Open Problems

**Conjecture 8.1** (Three-loop coefficient). *The three-loop coefficient in the additive form (2) is*

$$N_+ = \Omega_0 \alpha^{-1} \frac{\dim G_2 - h^\vee(A_2)}{h^\vee(G_2)} + O\left(\frac{\alpha}{\pi}\right),$$

where  $\Omega_0 = \pi^3/4$  and  $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ . Numerically:  $N_+ = 2921.178\dots$  vs. exact  $2920.838\dots$  (relative error 0.012%).

**Remark 8.1.** *The correction  $O(\alpha/\pi)$  needed to close the 0.012% gap is of order  $N_+ \cdot \alpha/\pi \approx 6.78$ , which is close to  $\alpha^{-1}/\pi^2 \approx 13.9$ ,  $\dim G_2/h^\vee(A_2) \approx 4.67$ , and  $\Omega_0 \cdot h^\vee(G_2)/h^\vee(A_2) \approx 10.34$ . None provides a definitive identification; this is the primary open problem.*

### Open problems.

1. Derive  $N_+ = \Omega_0 \alpha^{-1} \cdot \frac{11}{4}$  from a four-dimensional path integral on  $S^3$  with  $G_2$  holonomy, using Paper 30 (GON) and Paper 18 ( $\hat{O}$ ).
2. Identify the exact four-loop coefficient from  $\delta_4 = 8.81 \times 10^{-7}$ .
3. Determine whether  $\pi^{10}/32 = \Omega_0^2 \cdot 2\pi$  (the runner-up at 0.194%) is a coincidence or encodes a  $G_2$ -geometric identity  $\dim G_2 \rightarrow \pi^{10}/32 \Omega_0^{-2}/(2\pi)$ .
4. Extend the scan to  $E_6, E_7, E_8$  to test whether  $m_\tau/m_e$  and  $m_\tau/m_\mu$  admit similar  $\alpha^{-1}$ -enhanced representations.
5. Resolve the sign pattern: the series  $+1, -, +, +, \dots$  does not alternate; clarify whether this signals a finite radius of convergence in  $\alpha/\pi$ .

### Numerical summary.

$$N_+ = 2920.8380, \quad N_{\text{conj}} = 2921.178, \quad \delta_4 = 8.81 \times 10^{-7}, \quad (\text{Conjecture 8.1: PROMISING}).$$

Verification: `Lumen/corpus/addenda/verify/verify_P199.py` (50 assertions).