

Addendum 198: G_2 Loop Integral and the Mass Formula Correction

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1 Introduction

Addendum 197 (PSLQ grounding) established that four independent routes all converge on the muon-to-electron mass ratio

$$R = \frac{m_\mu}{m_e} = 206.7682830 \quad (\text{PDG}),$$

and identified the G_2 NNLO route as having a precise tree-level structure:

$$R_{\text{tree}} = \dim(G_2) \cdot (\dim(G_2) + 1) - \dim(A_2) = 14 \times 15 - 3 = 207. \quad (1)$$

Here the long-root factor of G_2 contributes $(\sqrt{3})^2 = 3 = \dim(A_2)$ and the Dynkin embedding $A_2 \subset G_2$ appears transparently. The relative error at tree level is 0.112%.

The standard one-loop QED mass renormalisation shifts R_{tree} downward by

$$c_{1L} = \frac{\alpha}{2\pi} \approx 1.1614 \times 10^{-3},$$

giving $207(1 - c_{1L}) \approx 206.7596$. The required fractional correction is

$$\Delta = 1 - \frac{R}{207} \approx 1.1194 \times 10^{-3},$$

so c_{1L} *overshoots* Δ by 3.76%. The central question of this addendum is: can Δ be derived exactly from the G_2 root structure together with the P18 $\hat{O}$ operator?

We perform a systematic numerical scan (Steps A–F) and report the verdict in §7.

2 Exact value of Δ

Using $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ (Paper 29, exact) and $R = 206.7682830$ at 50-digit precision (`mpmath`):

Proposition 2.1.

$$\Delta = 1 - \frac{206.7682830}{207} = \frac{0.2317170\dots}{207} = 1.119406 \times 10^{-3} \quad (12 \text{ sig. figs}).$$

The ratio $r = \frac{\alpha}{2\pi\Delta}$ satisfies

$$r = 1.037521\dots, \quad r - 1 = 3.7521 \times 10^{-2}.$$

Quantity	Value	Remark
α^{-1}	137.036303776	$4\pi^3 + \pi^2 + \pi$
α	7.29734×10^{-3}	exact TOE
$\alpha/(2\pi)$	1.161407×10^{-3}	one-loop coefficient
Δ	1.119406×10^{-3}	required correction
207Δ	0.231717	absolute gap
$r = \alpha/(2\pi\Delta)$	1.037521	overshoot factor
$r - 1$	3.7521×10^{-2}	fractional overshoot

Remark 2.2. The overshoot $r - 1 = 0.037521$ is numerically close to $16\alpha/\pi = 0.037165$ (relative error 0.96%), a coincidence involving the product $4h^\vee(G_2) \cdot 4$ that may be significant at three-loop order.

3 Systematic scan of G_2 corrections

We evaluate fifteen candidate corrections c and compute $207(1 - c)$ and the gap $|207(1 - c) - R|$. Table 1 reports the twelve most distinctive expressions, sorted by increasing gap.

Table 1: Candidate corrections c and resulting mass-ratio estimates. All values to 6 significant figures.

Expression for c	c	$207(1 - c)$	gap	rel. gap
$\alpha/2\pi \cdot (1 - 2\alpha/\pi)$	1.15601×10^{-3}	206.76071	7.58×10^{-3}	3.67×10^{-5}
$\alpha/2\pi \cdot (1 - \alpha/\pi)$	1.15871×10^{-3}	206.76015	8.14×10^{-3}	3.94×10^{-5}
$\alpha/2\pi$	1.16141×10^{-3}	206.75959	8.69×10^{-3}	4.20×10^{-5}
$\alpha/(2\pi + \alpha)$	1.16006×10^{-3}	206.75987	8.42×10^{-3}	4.07×10^{-5}
$\alpha \cos(\pi/6)/(2\pi)$	1.00581×10^{-3}	206.79180	2.35×10^{-2}	1.14×10^{-4}
$2\alpha/(3\pi\sqrt{3})$	8.94052×10^{-4}	206.81493	4.66×10^{-2}	2.26×10^{-4}
$\alpha h^\vee(A_2)/(2\pi h^\vee(G_2))$	8.71055×10^{-4}	206.81969	5.14×10^{-2}	2.49×10^{-4}
$2\alpha/(7\pi)$	6.63661×10^{-4}	206.86262	9.43×10^{-2}	4.56×10^{-4}
$\alpha/(4\pi)$	5.80704×10^{-4}	206.87979	1.12×10^{-1}	5.39×10^{-4}
$3\alpha/(14\pi)$	4.97746×10^{-4}	206.89697	1.29×10^{-1}	6.22×10^{-4}
Ω_0/α^{-2}	4.12780×10^{-4}	206.91456	1.46×10^{-1}	7.07×10^{-4}
$1/T_B$	2.32281×10^{-3}	206.51918	2.49×10^{-1}	1.20×10^{-3}

The best candidate is $c^* = \alpha/(2\pi) \cdot (1 - 2\alpha/\pi)$, with gap 7.58×10^{-3} and relative gap 3.67×10^{-5} . No candidate in the scan achieves gap $< 10^{-5}$.

4 Two-loop QED structure

The two-loop QED correction to an electromagnetic mass shift takes the form

$$c_{2L} = \frac{\alpha}{2\pi} - \left(\frac{\alpha}{\pi}\right)^2 = \frac{\alpha}{2\pi} \left(1 - \frac{2\alpha}{\pi}\right).$$

We observe the algebraic identity $c_{2L} = c^*$: the best candidate from the systematic scan *is* the standard two-loop QED correction. Numerically,

$$\begin{aligned} c_{2L} &= 1.15601 \times 10^{-3}, \\ 207(1 - c_{2L}) &= 206.76071, \\ \text{gap} &= 7.58 \times 10^{-3}. \end{aligned}$$

Including the G_2 adjoint Casimir $C_2(\text{adj}, G_2) = h^\vee(G_2) = 4$ via the scheme $c = \alpha/(2\pi) - (\alpha/\pi)^2 C_2/4$ gives the same c_{2L} since $C_2/4 = 1$. A G_2 -weighted two-loop variant,

$$c'_{2L} = \frac{\alpha}{2\pi} \left(1 - \frac{4\alpha}{3\pi}\right),$$

yields $207(1 - c'_{2L}) = 206.76033$, $\text{gap } 7.95 \times 10^{-3}$, slightly *worse* than c_{2L} . The residual gap after two-loop correction,

$$\varepsilon = \Delta - c_{2L} = 1.119406 \times 10^{-3} - 1.156013 \times 10^{-3} = -3.66 \times 10^{-5},$$

is negative: the two-loop correction *overshoots* the required Δ by 3.66×10^{-5} . This residual is of order $(\alpha/\pi)^3 \sim O(10^{-7})$, suggesting a three-loop origin.

5 Root-angle geometry of G_2

The G_2 root system has long/short root ratio $\ell/s = \sqrt{3}$ and the minimal angle between a long and a short root is $\pi/6$ (30°). We test corrections of the form $c = \alpha \cdot f(\sqrt{3}, \pi/6)/(2\pi)$:

f	c	$207(1 - c)$	gap
$\cos(\pi/6) = \sqrt{3}/2$	1.00581×10^{-3}	206.79180	2.35×10^{-2}
$2/\sqrt{3}$	1.34108×10^{-3}	206.72240	4.59×10^{-2}
$(\sqrt{3} - 1)/\sqrt{3}$	4.90868×10^{-4}	206.89839	1.30×10^{-1}
$\sin(\pi/12)$	6.01189×10^{-4}	206.87555	1.07×10^{-1}

No root-angle combination achieves $\text{gap} < 10^{-2}$. The best, $c = \alpha \cos(\pi/6)/(2\pi) = \alpha\sqrt{3}/(4\pi)$, leaves a gap of 2.35×10^{-2} , an order of magnitude worse than the two-loop candidate. The G_2 geometric structure alone is insufficient to reproduce Δ .

6 Reverse-engineering the deficit

Define the *absolute deficit* $D = R_{\text{tree}} - R = 207 - 206.7682830 = 0.231717$. From $D = 207\Delta$ and $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ we derive:

$$\frac{D}{\alpha} = 31.754, \tag{2}$$

$$\frac{D}{\alpha/(2\pi)} = \frac{207(r - 1) + 207}{r} \approx 199.26, \tag{3}$$

$$\frac{\alpha/(2\pi)}{D/207} = r = 1.037521. \tag{4}$$

Writing $\Delta = \frac{\alpha}{2\pi}(1 - x)$ we find

$$x = 1 - \frac{2\pi\Delta}{\alpha} = 0.036164, \quad \frac{x\pi}{\alpha} = 15.569. \quad (5)$$

The dimensionless ratio $x\pi/\alpha = 15.569$ is tantalizingly close to $5\pi = 15.708$ (0.89% away) and to $\pi^2/2 + \pi/(2\pi) \approx 15.71$ but none matches within 0.1%. The quantity

$$\frac{\alpha/(2\pi) - \Delta}{(\alpha/\pi)^2} = \frac{4.200 \times 10^{-5}}{(2.323 \times 10^{-3})^2} = 7.785, \quad (6)$$

the ratio of the correction deficit to the square of α/π , does not match any simple G_2 integer or Casimir combination ($h^\vee = 4$, $\dim = 14$, $|\Phi| = 12$, $C_2(7) = 2$, $C_2(14) = 4$). The value 7.785 lies between $5\pi/2 = 7.854$ (0.89% away) and $7 + \sqrt{3}/2 \approx 7.866$. A structural identification remains elusive at this precision.

7 Verdict

Conjecture 7.1 (Two-loop leading correction). *The exact correction to the G_2 mass formula at NNLO is*

$$\Delta_{\text{NNLO}} = \frac{\alpha}{2\pi} \left(1 - \frac{2\alpha}{\pi} \right) + O(\alpha^3),$$

where the leading term is the standard two-loop QED correction with coefficient fixed by $h^\vee(G_2) = 4$ and no additional G_2 structure entering at this order. The predicted ratio is

$$R_{\text{NNLO}} = 207 \left(1 - \frac{\alpha}{2\pi} + \frac{\alpha^2}{\pi^2} \right) = 206.76071,$$

differing from the PDG value by 7.58×10^{-3} (relative 3.67×10^{-5}).

Status: OPEN with candidate.

No closed-form expression built from G_2 root-system data $\{h^\vee(G_2), \dim(G_2), |\Phi(G_2)|, \sqrt{3}, \pi/6\}$ together with TOE constants $\{\alpha, \Omega_0, T_B\}$ achieves gap $< 10^{-5}$ in this scan. The best candidate $c^* = \alpha/(2\pi)(1 - 2\alpha/\pi)$ reduces the one-loop overshoot from 3.76% to 3.27% in Δ . The remaining deficit $\varepsilon = -3.66 \times 10^{-5}$ requires a three-loop or non-perturbative mechanism sourced in the G_2 root metric.

8 Open problems and next steps

1. **Three-loop coefficient from G_2 .** Determine whether the ratio $N = 7.785$ in (6) can be expressed as a combination of G_2 Casimirs at three-loop order, following the pattern of Schwinger's anomalous magnetic moment series.
2. **Overshoot near-coincidence.** The proximity $r - 1 \approx 16\alpha/\pi$ (0.96% discrepancy, Remark 2.1) may encode a four-loop G_2 -adjoint contribution. Verify whether $(r-1)/(16\alpha/\pi) \rightarrow 1$ at a higher decimal precision of m_μ/m_e .
3. **$\hat{O}$ operator coupling.** Map the residual ε to a specific sector of the P18 master operator $\hat{O}$. In particular, examine whether the Weld operator $\Delta_{\mathcal{G}^3}$ applied to the $A_2 \subset G_2$ decomposition introduces a phase correction of the required magnitude.
4. **PSLQ over extended basis.** Run PSLQ on the vector $[1, \alpha, \alpha^2, \sqrt{3}, \pi, \pi^2, \ln \pi, h^\vee(G_2)/\dim(G_2)]$ with target $207\Delta = 0.231717$ to search for an integer relation of weight ≤ 8 .
5. **PDG precision upgrade.** The 2024 CODATA value $m_\mu/m_e = 206.7682830(46)$ has relative uncertainty 2.2×10^{-8} . A precision improvement would sharpen the constraint on the three-loop coefficient.