

SR3's Remaining Open Step:
Uniqueness of the Exponent in $\zeta = \alpha^p$,
and Independence of the Mass Formula Gap

Addendum 196 to the TOE Corpus

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Abstract

Addendum P191 closed SR3 for the coefficients α and β , showed $\gamma = 3/4$ is geometrically forced (pending the P10 five-fold convergence uniqueness theorem), and partially closed $\zeta = \alpha^{5/4}$ via the dimensional argument $5/4 = 1 + 1/\dim(B^4)$ confirmed by P184's E_6/F_4 Killing norm matching. P191 §6 identified the uniqueness of the exponent $5/4$ as the remaining open step within SR3. This addendum attacks that step directly.

We first write the within-family mass ratio as an explicit function of the exponent p : $m_{\text{ratio}}(p; \ell_0, \ell_1) = \exp(\mu_1/\mu_0 \cdot \Delta E(p; \ell_0, \ell_1))$, where ΔE is monotone decreasing in p for every pair (ℓ_0, ℓ_1) with $\ell_1 > \ell_0$. A numerical scan over $p \in \{1, \frac{5}{4}, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5\}$ and canonical pairs reveals a *structural gap*: the target $m_\mu/m_e \approx 206.77$ is unreachable for any $p \geq 0$ at any canonical pair. Specifically, the range of $m_{\text{ratio}}(p; 1, 2)$ over $p \in (0, \infty)$ is $(50.21, 110.99)$, the range of $m_{\text{ratio}}(p; 2, 3)$ is $(249.04, \infty)$, and 206.77 falls in the gap $(110.99, 249.04)$ not covered by any pair.

We then assess the P184 E_6/F_4 Killing norm argument with precision: P184 Theorem 5.1 is tight for the *direction* (the Hopf geometry uniquely selects $H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$) but does not independently fix the *magnitude* $p = 5/4$. The Killing norm check $|\zeta \cdot H_{U(1)}|^2 = \alpha^{5/2} \times 24$ is a consistency verification, not an independent derivation. The genuine source of $p = 5/4$ is P16's dimensional counting: $p = 1 + \dim(S^1)/\dim(B^4) = 1 + 1/4$. We identify what a formal uniqueness proof would require and state it as a conditional theorem.

Theorem A (unconditional): For all $p \geq 0$ and all canonical level pairs (ℓ_0, ℓ_1) with $0 \leq \ell_0 < \ell_1 \leq 4$, $m_{\text{ratio}}(p; \ell_0, \ell_1) \neq 206.77$. The mass formula gap is independent of the exponent question.

Theorem B (conditional): If the P16 dimensional counting formula $p = 1 + \dim(S^1)/\dim(B^4)$ is formalised as a proposition derivable from the Hopf fibration degree of the boundary restriction operator, then $p = 5/4$ is the unique exponent consistent with the E_6/F_4 Killing norm and SR3 is fully closed.

Numerical companion: `verify/verify_P196.py` (mp.dps=55; 39/39 checks pass, exit 0).

The ζ Exponent in the Assembly

1.1 Position in SR3

Paper 18 [2] assembles the canonical master operator

$$\hat{O} = D_{B^4}^2 + \Delta_{S^3} + \Delta_{S^1} + V_{\text{self}}(r) + \alpha\rho(r) + \gamma T + \zeta\mathcal{R} + \beta\mathcal{M}, \quad (1)$$

with four coefficients

$$\alpha = \frac{1}{\alpha^{-1}} \approx \frac{1}{137.036}, \quad \gamma = \frac{3}{4}, \quad \zeta = \alpha^{5/4}, \quad \beta = \frac{3\pi}{20}. \quad (2)$$

The assembly-uniqueness question (P18-T2) decomposes into three sub-results. SR2 is closed (P186: R8 forces additive composition). SR3 addresses whether $(\alpha, \gamma, \zeta, \beta)$ are the only spectrum-matching coefficients. P191 showed SR3 is partially closed: α and β are uniquely forced by linear spectral conditions (closed); $\gamma = 3/4$ is geometrically forced pending P10's five-fold convergence uniqueness (partially closed); $\zeta = \alpha^{5/4}$ is derived from the dimensional formula and P184's Killing norm confirmation, but the uniqueness of the exponent $5/4$ was recorded as an open step.

Assertion 1 (P191 open step for ζ). P191 Open Problem 6(ii) records: “*State as a formal proposition that the Killing norm matching in P184 Theorem 5.1 forces the exponent 5/4 in $\zeta = \alpha^{5/4}$ as the unique value consistent with the graded E_6/F_4 structure and the Hopf $U(1)$ normalisation.*” This is the remaining open step within SR3. The present paper addresses it.

1.2 What is established about $\zeta = \alpha^{5/4}$

Three independent sources converge on $\zeta = \alpha^{5/4}$.

Assertion 2 (P18 coefficient determination). P18 Proposition 4.2 states: the renormalization-generator coefficient is $\zeta = \kappa = \alpha^{5/4}$ where the oscillation scale κ is derived in Paper 16 from the dimensional analysis $5/4 = 1 + 3/(4 \times 3)$.

Assertion 3 (P184 formal bridge). P184 Theorem 5.1 proves: $\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta \cdot H_{U(1)}$, where $H_{U(1)}$ is the Hopf right- $U(1)$ generator in the F_4 -centralising complement $\mathfrak{z} \subset \mathfrak{h}_{E_6}$. The direction is uniquely fixed by Hopf geometry; the coefficient $\zeta = \alpha^{5/4}$ is imported from P18. P184 Assertion 7.3 states explicitly: “*The coefficient $\zeta = \alpha^{5/4}$ is a positive scalar. It rescales the generator $H_{U(1)}$ but does not change which direction in $\mathfrak{z}$ is selected.*”

Assertion 4 (Dimensional formula). The exponent 5/4 has the geometric decomposition:

$$\frac{5}{4} = 1 + \frac{1}{\dim(B^4)} = 1 + \frac{\dim(S^1)}{\dim(B^4)} = \underbrace{1}_{\text{fibre coupling}} + \underbrace{\frac{1}{4}}_{\text{bulk reciprocal}}, \quad (3)$$

encoding the fibre dimension $\dim(S^1) = 1$ and the reciprocal bulk dimension $1/\dim(B^4) = 1/4$. P18 records the equivalent form $5/4 = 1 + 3/(4 \times 3)$, where $3 = \dim(S^3)$ (boundary dimension). Both are topological invariants of the Hopf triple $S^1 \hookrightarrow S^3 \rightarrow S^2$.

The key gap identified by P191: *are there other values of p that are equally consistent with the E_6/F_4 structure?* Section 4 shows that P184’s argument pins the direction but not the magnitude, and the dimensional formula is the genuine source of $p = 5/4$.

2 Mass Formula as a Function of p

2.1 The canonical within-family mass ratio

From P18 §5.4 (Mass Spectrum Conjecture) and P191 §2.3, the within-family mass ratio between levels ℓ_0 and ℓ_1 (same family index k) is

$$m_{\text{ratio}}(p; \ell_0, \ell_1) = \exp\left(\frac{\mu_1}{\mu_0} \cdot \Delta E(p; \ell_0, \ell_1)\right), \quad (4)$$

where the energy gap is

$$\Delta E(p; \ell_0, \ell_1) = \underbrace{[\ell_1(\ell_1 + 2) - \ell_0(\ell_0 + 2)]}_{\Delta E_{S^3}} + \underbrace{\alpha^p \cdot (\ell_1 - \ell_0)}_{\zeta \text{ contribution}} + \underbrace{\beta \frac{\mu_{\ell_1} - \mu_{\ell_0}}{\mu_0}}_{\beta \text{ contribution}}. \quad (5)$$

The γT contribution cancels in within-family ratios (P191 §2.3, Remark 3.4).

Remark 2.1 (Constants). $\alpha^{-1} = \mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036$, $\mu_1 = 16\pi^3/5 + 3\pi^2/4 + 2\pi/3 \approx 108.717$, $\mu_1/\mu_0 \approx 0.7933$, $\beta = 3\pi/20 \approx 0.47124$. The moments $\mu_n = 16\pi^3/(n+4) + 3\pi^2/(n+3) + 2\pi/(n+2)$ are strictly decreasing (verified: Check 2e of `verify_P196.py`).

2.2 Properties of $m_{\text{ratio}}(p; \ell_0, \ell_1)$

Proposition 2.2 (Monotonicity in p). *For every pair (ℓ_0, ℓ_1) with $\ell_1 > \ell_0$, the function $p \mapsto m_{\text{ratio}}(p; \ell_0, \ell_1)$ is strictly decreasing on $(0, \infty)$.*

Proof. Since $\alpha < 1$, the function $p \mapsto \alpha^p$ is strictly decreasing and positive for all $p \in \mathbb{R}$. The term $\alpha^p(\ell_1 - \ell_0) > 0$ is therefore strictly decreasing in p . The remaining terms in ΔE do not depend on p . Hence ΔE is strictly decreasing in p , and $m_{\text{ratio}} = \exp((\mu_1/\mu_0) \cdot \Delta E)$ is strictly decreasing (Checks 5a–5c of `verify_P196.py`). $\square$

Proposition 2.3 (Asymptotic limits). *For each pair (ℓ_0, ℓ_1) with $\ell_1 > \ell_0$, define $\Delta E_\infty = \Delta E_{S^3} + \beta(\mu_{\ell_1} - \mu_{\ell_0})/\mu_0$ (the limit as $p \rightarrow \infty$, $\zeta \rightarrow 0$) and $\Delta E_0 = \Delta E_\infty + (\ell_1 - \ell_0)$ (the limit as $p \rightarrow 0^+$, $\zeta \rightarrow 1$). Then:*

$$\lim_{p \rightarrow \infty} m_{\text{ratio}}(p; \ell_0, \ell_1) = \exp\left(\frac{\mu_1}{\mu_0} \cdot \Delta E_\infty\right) =: R_\infty(\ell_0, \ell_1), \quad (6)$$

and m_{ratio} approaches (but does not reach) $R_0(\ell_0, \ell_1) := \exp((\mu_1/\mu_0) \cdot \Delta E_0)$ as $p \rightarrow 0^+$. For $p \in (0, \infty)$:

$$m_{\text{ratio}}(p; \ell_0, \ell_1) \in (R_\infty(\ell_0, \ell_1), R_0(\ell_0, \ell_1)). \quad (7)$$

Proof. Immediate from Proposition 2.2 and the limits $\lim_{p \rightarrow \infty} \alpha^p = 0$, $\lim_{p \rightarrow 0^+} \alpha^p = 1$. $\square$

2.3 Target for m_μ/m_e

The experimental ratio is $m_\mu/m_e \approx 206.77$ (CODATA 2018: $m_\mu/m_e = 206.7682830$). The required energy gap for this target is

$$\Delta E_* = \frac{\ln(206.77)}{\mu_1/\mu_0} = \frac{5.332}{0.7933} \approx 6.722. \quad (8)$$

3 Numerical Scan: p Versus Mass Ratio

3.1 Scan table

Table 1 shows $m_{\text{ratio}}(p; \ell_0, \ell_1)$ for three representative pairs and eight values of p , computed at `mp.dps = 55`.

Remark 3.1 (Rapid convergence of ζ to zero). For $p \geq 2$, the contribution $\zeta = \alpha^p < 5.3 \times 10^{-5}$ is negligible: the table entries stabilise to four-digit agreement with the $p \rightarrow \infty$ limits. The observable range of m_{ratio} as p varies over $(0, \infty)$ is determined almost entirely by the S^3 eigenvalue structure, not by the precise value of p .

3.2 Structural gap

Table 1 reveals the central finding of this paper: no value of $p \in \{1, \frac{5}{4}, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5\}$, and indeed no $p \geq 0$, produces $m_{\text{ratio}}(p; \ell_0, \ell_1) \approx 206.77$ for any of the three canonical pairs. The column for pair (2, 3) is consistently above 249; the column for pair (1, 2) is consistently near 50.

Proposition 3.2 (Range bounds). *For $p \in (0, \infty)$:*

Table 1: Mass ratios $m_{\text{ratio}}(p; \ell_0, \ell_1)$ vs. exponent p . Target: 206.77. The canonical value is $p = 5/4$ (bold row). Pair (1, 2) is the “nearest below” pair; pair (2, 3) is the “nearest above” pair. The target falls in the structural gap between them.

p	$\zeta = \alpha^p$	ratio(0, 1)	ratio(1, 2)	ratio(2, 3)
1	7.297×10^{-3}	10.060	50.497	250.49
5/4	2.133×10^{-3}	10.019	50.291	249.46
3/2	6.234×10^{-4}	10.007	50.231	249.17
2	5.325×10^{-5}	10.002	50.208	249.05
5/2	4.549×10^{-6}	10.002	50.206	249.04
3	3.886×10^{-7}	10.002	50.206	249.04
4	2.836×10^{-9}	10.002	50.206	249.04
5	2.069×10^{-11}	10.002	50.206	249.04
<i>target</i>			$\longleftarrow 206.77 \longrightarrow$	

(a) $m_{\text{ratio}}(p; 1, 2) \in (50.21, 111.0)$ (infimum at $p \rightarrow \infty$, supremum as $p \rightarrow 0^+$).

(b) $m_{\text{ratio}}(p; 2, 3) \in (249.0, \infty)$ (infimum at $p \rightarrow \infty$, no finite supremum).

(c) $m_{\text{ratio}}(p; 0, 1) \in (10.00, 22.1)$.

(d) $m_{\text{ratio}}(p; 0, 2) \in (502, \infty)$.

In particular, $206.77 \in (111.0, 249.0)$ falls in the structural gap between the supremum of range (a) and the infimum of range (b). No canonical pair covers this gap for any $p \geq 0$.

Proof. Direct computation from Proposition 2.3 using the values in Table 2. Verified to 55-digit precision in Checks 6a–6f of `verify_P196.py`. $\square$

Table 2: Asymptotic limits R_∞ and R_0 for canonical pairs.

Pair (ℓ_0, ℓ_1)	ΔE_∞	ΔE_0	$R_\infty (p \rightarrow \infty)$	$R_0 (p \rightarrow 0^+)$
(0, 1)	2.9026	3.9026	10.002	22.11
(1, 2)	4.9362	5.9362	50.21	111.0
(2, 3)	6.9549	7.9549	249.0	550.6
(0, 2)	7.8388	9.8388	502	2449

3.3 Values of p^* for which $m_{\text{ratio}} = 206.77$

For each pair, we solve $m_{\text{ratio}}(p; \ell_0, \ell_1) = 206.77$ for p . From equation (5): the required $\zeta^* = (\Delta E_* - \Delta E_\infty)/(\ell_1 - \ell_0)$ and $p^* = \log \zeta^* / \log \alpha$.

For pairs (0, 1) and (1, 2): the solution exists but requires $p^* < 0$, i.e. $\zeta^* = \alpha^{p^*} > 1$, which is unphysical ($\zeta \approx 2.1 \times 10^{-3}$ in the physical assembly and must satisfy $\zeta \ll 1$). For pairs (2, 3) and (0, 2): the S^3 eigenvalue gap alone already exceeds the target $\Delta E_* = 6.722$, so no real solution exists. (Verified in Checks 7a–7h of `verify_P196.py`.)

Table 3: Values of p^* satisfying $m_{\text{ratio}}(p^*; \ell_0, \ell_1) = 206.77$. All values are unphysical: either $p^* < 0$ (so $\zeta^* > 1$, violating the perturbativity requirement $\zeta \ll 1$) or $\zeta^* < 0$ (impossible).

Pair (ℓ_0, ℓ_1)	ΔE_∞	$\zeta^* = \Delta E_* - \Delta E_\infty$	p^*	Physical?
(0, 1)	2.9026	3.819	-0.272	No ($\zeta^* > 1, p^* < 0$)
(1, 2)	4.9362	1.786	-0.118	No ($\zeta^* > 1, p^* < 0$)
(2, 3)	6.9549	-0.233	—	No ($\zeta^* < 0$, impossible)
(0, 2)	7.8388	-1.117/2	—	No ($\zeta^* < 0$, impossible)

4 E_6 Uniqueness Analysis: Does P184 Close It?

4.1 What P184 proves

P184 Theorem 5.1 (the formal bridge theorem) establishes:

$$\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta \cdot H_{U(1)}, \quad H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}. \quad (9)$$

The proof chains: (i) the holomorphic Euler decomposition of $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ on $\mathbb{C}^2 \supset B^4$ separates the radial component (normal to S^3) from the Hopf fibre component $\partial/\partial\psi$; (ii) P176 Theorem 4.4 identifies $\partial/\partial\psi = H_{U(1)}$, the F_4 -centralising Cartan generator with Killing norm squared $|H_{U(1)}|^2 = 24$ (P182: $24 = 2|\Phi_{G_2}|$).

Assertion 5 (P184 direction uniqueness — tight). The choice of $H_{U(1)}$ as the direction in $\mathfrak{z}$ is uniquely forced:

- (i) The centraliser $\mathfrak{z} \subset \mathfrak{h}_{E_6}$ is 2-dimensional (P176 Proposition 4.1: $\text{rank}(E_6) - \text{rank}(F_4) = 2$).
- (ii) The Hopf fibration on S^3 is unique up to orientation. The Hopf right- $U(1)$ action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$ commutes with the F_4 left-action on $\text{Sp}(1) \cong S^3$, so its generator lies in $\mathfrak{z}$.
- (iii) The orthogonal direction $H_{U(1)}^\perp = \partial/\partial\phi$ (the S^2 azimuthal rotation, P188) vanishes at the Hopf poles $\chi = 0, \pi$ (P189 Theorem 4.1: Hopf Regularity Axiom), excluding it from the assembly. $H_{U(1)} = \partial/\partial\psi$ is nowhere-vanishing.

Hence the direction $H_{U(1)} \in \mathfrak{z}$ is uniquely forced by the Hopf fibration geometry, independently of the magnitude ζ .

4.2 What P184 does not prove: the magnitude

Assertion 6 (P184 magnitude import). P184 Assertion 7.3 states explicitly: “The particular value $\zeta = \alpha^{5/4}$ is fixed by the P18 coefficient-determination theorem (P18 Proposition 4.2) and encodes the renormalization group scale.” P184 does not derive $p = 5/4$ independently from the E_6/F_4 Killing norm. The consistency check is:

$$|\zeta \cdot H_{U(1)}|^2 = \zeta^2 \times 24 = \alpha^{5/2} \times 24, \quad (10)$$

which is satisfied when $\zeta = \alpha^{5/4}$. But equation (10) does not independently constrain p : any value $\zeta = \alpha^p$ gives $|\zeta H_{U(1)}|^2 = \alpha^{2p} \times 24$, and the constraint merely reflects that $p = 5/4$ once ζ is taken as $\alpha^{5/4}$ from P18.

4.3 The Killing norm as consistency check vs. independent derivation

To close the uniqueness gap, the Killing norm matching must impose an *external* constraint on p , not merely verify a substitution. Three approaches are available.

Approach A (Dual Coxeter ratio). The dual Coxeter ratio $h^\vee(E_6)/h^\vee(F_4) = 12/9 = 4/3$ is the natural E_6/F_4 algebraic invariant. Setting $\alpha^p = 4/3$ gives $p = \ln(4/3)/\ln(\alpha^{-1}) \approx 0.0585$, which is *not* $5/4$. The Coxeter ratio as a direct exponent source is excluded.

Approach B (Dimensional coupling). The renormalization generator $\mathcal{R}$ acts on B^4 (dimension 4) and its boundary restriction lives on the Hopf fibre S^1 (dimension 1). The natural coupling of $\mathcal{R}$ to the fine-structure constant α is: one power of α for the intrinsic scale of $\mathcal{R}$ (a first-order operator on B^4 , contributing exponent 1); one additional power $\alpha^{1/\dim(B^4)} = \alpha^{1/4}$ for the boundary restriction ($B^4 \rightarrow S^3$ projection reduces the effective dimension by $\dim(B^4) = 4$, contributing the reciprocal). The total coupling is $\alpha^{1+1/4} = \alpha^{5/4}$, giving $p = 5/4$. This is the argument recorded in P16 and cited in P18 and P191.

Approach C (Hopf fibration degree). A formal proposition (the candidate Killing norm uniqueness theorem) would state: the unique coupling $\zeta = \alpha^p$ such that $\zeta \cdot H_{U(1)}$ carries Hopf fibration degree $\deg(\pi) = 1$ at the α -scale is $\zeta = \alpha^{5/4}$. This route derives $p = 5/4$ from the Hopf invariant and the bulk dimension without appealing to P16’s dimensional counting.

Remark 4.1 (Status of the three approaches). Approach A is excluded (wrong prediction: $p \approx 0.06 \neq 5/4$). Approach B is the actual P16 derivation; it can be converted into a formal proposition by specifying how “fibre coupling exponent 1” and “bulk reciprocal exponent $1/4$ ” are computed from the Hopf geometry. Approach C is the algebraic route through the Hopf invariant; if made precise, it would provide an independent algebraic derivation of $p = 5/4$ within the E_6/F_4 framework.

4.4 Summary: what P184 closes and what remains

Table 4: Decomposition of the ζ uniqueness claim.

Component	Status	Evidence
Direction: $H_{U(1)} \in \mathfrak{z}$	Closed (P184)	Hopf geometry, P176 Thm 4.4
Magnitude: $p = 5/4$	Partially closed	P16 (dimensional), P18
Formal uniqueness of p	Open step	Requires formal proposition

The gap is precise: P184 proves the direction uniquely; P16 derives the magnitude via dimensional counting. Full formal closure requires converting P16’s dimensional argument into a proposition derivable from the Hopf fibration geometry alone, without ad hoc dimensional counting. This is a well-defined task (Approach C above) requiring no new corpus papers but only formalising the Hopf fibration degree argument.

5 Theorem: SR3 Status for ζ

Theorem 5.1 (Structural exclusion — unconditional). *For all $p \geq 0$ and all canonical level pairs (ℓ_0, ℓ_1) with $0 \leq \ell_0 < \ell_1 \leq 4$:*

$$m_{\text{ratio}}(p; \ell_0, \ell_1) \neq 206.77. \quad (11)$$

Equivalently, the mass formula gap is independent of the exponent p and cannot be closed by varying p within the physical range $p > 0$.

Proof. We partition the canonical level pairs by the sign of $\Delta E_\infty - \Delta E_*$, where $\Delta E_* = 6.722$ (equation (8)).

Below pairs ($\Delta E_\infty < \Delta E_*$): pairs $(0, 1)$ and $(1, 2)$.

The range of $m_{\text{ratio}}(p; \ell_0, \ell_1)$ over $p \in (0, \infty)$ is (R_∞, R_0) (Proposition 3.2). We verify $R_0 < 206.77$ (the supremum, approached as $p \rightarrow 0^+$, is below target):

$$R_0(1, 2) = \exp(0.7933 \times (4.936 + 1)) = \exp(4.706) \approx 111.0 < 206.77, \quad (12)$$

$$R_0(0, 1) = \exp(0.7933 \times (2.903 + 1)) = \exp(3.096) \approx 22.1 < 206.77. \quad (13)$$

Hence $m_{\text{ratio}}(p; \ell_0, \ell_1) < 206.77$ for all $p \in (0, \infty)$ for these pairs.

Above pairs ($\Delta E_\infty \geq \Delta E_*$): pairs $(2, 3)$, $(0, 2)$, and all higher pairs.

For these pairs $R_\infty \geq 249.0 > 206.77$ and m_{ratio} is decreasing in p , so $m_{\text{ratio}}(p; \ell_0, \ell_1) \geq R_\infty > 206.77$ for all $p \in (0, \infty)$.

At $p = 0$: the boundary value $\zeta = \alpha^0 = 1$ is unphysical, but for completeness: below pairs remain below 206.77 (equations (12)–(13)); above pairs are even larger.

Gap location: The range of m_{ratio} over all canonical pairs and all $p \geq 0$ contains neither 206.77 nor any value in the interval $(111.0, 249.0)$. The target $206.77 \in (111.0, 249.0)$ is structurally excluded.

Numerical verification: Check 8a of `verify_P196.py` performs an exhaustive scan over all pairs $0 \leq \ell_0 < \ell_1 \leq 4$ and $p \in \{0, 0.25, 0.50, \dots, 10.0\}$ (41 p -values) and confirms no pair comes within 1.0 of the target. $\square$

Corollary 5.2 (Mass formula gap is orthogonal to ζ exponent question). *The gap between the computed ratio ≈ 50 (pair $(1, 2)$, canonical p) and the experimental ratio ≈ 206.77 cannot be closed by any choice of p . Resolving the ζ exponent uniqueness question (closing SR3 for ζ) would not repair the mass formula gap. The two questions are orthogonal.*

Theorem 5.3 (Conditional uniqueness of $p = 5/4$). *Suppose the following formal proposition holds:*

Proposition 5.4 (Hopf degree coupling). *The unique coupling $\zeta = \alpha^p$ such that $\zeta\mathcal{R}$, when restricted to the Hopf fibre $S^1 \subset S^3 = \partial B^4$, carries coupling exponent equal to $\dim(S^1) + 1/\dim(B^4) = 1 + 1/4 = 5/4$ is $p = 5/4$.*

Then $p = 5/4$ is the unique exponent consistent with the E_6/F_4 Killing norm and the Hopf fibration data, and SR3 is fully closed for ζ .

Proof. If Proposition 5.4 holds, then $p = 5/4$ is the unique solution of the Hopf degree constraint. Combined with the P184 uniqueness of the direction (Assertion 5), the complete operator $\zeta\mathcal{R} = \alpha^{5/4}\mathcal{R}$ is uniquely determined. The P18-T3 and P18-T4 spectral conditions fix α and β (P191 Theorems 4.1–4.2); the geometric dimension ratio fixes γ (P191 Theorem 4.3). All four coefficients being uniquely determined closes SR3. $\square$

Remark 5.5 (Status of Proposition 5.4). Proposition 5.4 is not yet proved in the corpus. It is the formalised version of P16’s dimensional counting argument ($p = 1 + \dim(S^1)/\dim(B^4)$, which P18 records as the source of $\zeta = \alpha^{5/4}$). The proposition requires specifying: (a) the “coupling exponent” of a differential operator on B^4 with respect to α , in the sense of the E_6/F_4 -graded Sobolev scale on B^4 ; (b) showing that $\dim(S^1) + 1/\dim(B^4)$ is the unique exponent consistent with the Hopf fibration degree $\deg(\pi) = 1$. This is a well-defined mathematical task requiring no new corpus papers.

5.1 SR3 status update

Table 5: SR3 coefficient uniqueness status after P196.

Coeff.	Value	Prior status (P191)	P196 contribution	Updated status
α	$1/\alpha^{-1}$	Closed	Confirmed	Closed
β	$3\pi/20$	Closed	Confirmed	Closed
γ	$3/4$	Partially closed	Not addressed	Partially closed
ζ	$\alpha^{5/4}$	Partially closed	Direction closed; magnitude gap characterised	Partially closed

The key advance of P196 relative to P191 for ζ : P196 confirms that P184 *tightly closes* the direction sub-question (no gap there, Assertion 5), characterises the magnitude gap precisely (requires formalising the Hopf degree coupling, Proposition 5.4), and proves unconditionally that the mass formula gap is independent of p (Theorem 5.1).

6 Conclusion

The remaining open step in SR3 is the formal uniqueness of the exponent $p = 5/4$ in $\zeta = \alpha^p$. This paper has:

- (i) Written the mass formula as an explicit function of p (equations (4)–(5)) and proved it is monotone decreasing in p for every canonical level pair (Proposition 2.2).
- (ii) Performed a numerical scan over $p \in \{1, \frac{5}{4}, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5\}$ and canonical pairs (Table 1), revealing a *structural gap*: the target $m_\mu/m_e \approx 206.77$ falls in $(111.0, 249.0)$, not covered by any canonical pair for $p \geq 0$.
- (iii) Proved **unconditionally** (Theorem 5.1) that no exponent $p \geq 0$ produces $m_{\text{ratio}} \approx 206.77$: the mass formula gap is independent of the exponent uniqueness question. Corollary 5.2: the two questions are orthogonal.
- (iv) Assessed P184’s E_6/F_4 argument with precision: the *direction* ($H_{U(1)} \in \mathfrak{z}$) is tightly closed by Hopf geometry; the *magnitude* ($p = 5/4$) is supported by P16’s dimensional argument but requires formalisation as a Hopf degree coupling proposition (the dual Coxeter ratio route gives $p \approx 0.06$, not $5/4$, and is excluded).
- (v) Stated **conditionally** (Theorem 5.3) that if Proposition 5.4 holds, then $p = 5/4$ is unique and SR3 is fully closed for ζ .

What is unconditional. Theorem 5.1 and Corollary 5.2 do not depend on any open step.

What is conditional. Theorem 5.3 depends on Proposition 5.4, not yet proved in the corpus. It is a formalisation of P16’s dimensional counting, requiring the notion of “coupling exponent” in the E_6/F_4 -graded Sobolev scale on B^4 .

SR3 summary. Three coefficients are closed or fully pinned: α (closed), β (closed), γ (pending P10 five-fold convergence theorem). The coefficient ζ is partially closed: the direction ($H_{U(1)}$) is tightly closed by P184, and the magnitude ($p = 5/4$) awaits the Hopf degree coupling proposition. The two remaining open steps in SR3 are of comparable tractability — both require formalising an existing geometric argument, not introducing new machinery.

Numerical Companion

All numerical claims are verified by `Lumen/corpus/addenda/verify/verify_P196.py` (mp.dps = 55; 39/39 checks pass, exit 0).

Key verified values:

- Check 4: canonical ratios at $p = 5/4$: pair $(0, 1) = 10.019$, pair $(1, 2) = 50.291$, pair $(2, 3) = 249.464$.
- Check 6: range bounds: pair $(1, 2) \in (50.21, 111.0)$, pair $(2, 3) \in (249.0, \infty)$, gap $(111.0, 249.0)$ contains 206.77.
- Check 7: $p^*(0, 1) = -0.272$, $p^*(1, 2) = -0.118$ (both negative); pairs $(2, 3)$, $(0, 2)$, $(1, 3)$ have no real p^* .
- Check 8: exhaustive scan, 41 p -values $\times$ 10 pairs, no hit within 1.0 of target.
- Check 9: $|H_{U(1)}|^2 = 24$ (P182); $|\zeta H_{U(1)}|^2 = \alpha^{5/2} \times 24$.
- Check 10: $p_{\dim} = 1 + \dim(S^1)/\dim(B^4) = 5/4$ (all three forms).

References

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- [3] L. F. Vlegels, *Addendum 176: Perturb as the $U(1)$ Cartan Generator in $E_6 \rightarrow F_4 \times U(1)$* , TOE Corpus, May 2026.
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- [5] L. F. Vlegels, *Addendum 184: OP-H Formal Derivation — Perturb = $H_{U(1)}$ via the Hopf Fibre Restriction*, TOE Corpus, May 2026.
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