

# Addendum 194: Mass Identification Rule Alternatives

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May 2026

## Abstract

Paper 18 maps  $m_e \leftrightarrow \lambda_1 = 3$  and requires  $\lambda_\ell \approx 620.3$  for the muon; P191 proved no integer solution exists (nearest miss  $\lambda_{24} = 624$ , gap 0.6%). We test seven alternative identification rules against the  $S^3$  spectrum and TOE constants. Six rules are definitively closed. One near-miss survives: the fifth eigenvalue  $\lambda_5 = 35$  satisfies  $\lambda_5^{3/2} = 35\sqrt{35} \approx 207.063$ , within 0.143% of  $m_\mu/m_e \approx 206.768$ . P18-Conj1 is left open with a single spectrally motivated candidate requiring  $t_\mu/t_e = \lambda_5$ .

## 1 Setup

$S^3$  spectrum:  $\lambda_\ell = \ell(\ell + 2)$ ,  $d_\ell = (\ell + 1)^2$ . Key values:  $\lambda_1=3$ ,  $\lambda_2=8$ ,  $\lambda_5=35$ ,  $\lambda_{19}=399$ ,  $\lambda_{24}=624$ . TOE constants (frozen in `kernel/math/quat_s3.py`):

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036, \quad T_\bullet = \pi\alpha^{-1} \approx 430.51, \quad \omega_0 = \frac{\pi^3}{4} \approx 7.752. \quad (1)$$

Target:  $R^* = m_\mu/m_e = 105.6584/0.51100 \approx 206.768$ .

## 2 Rule 1 — Density-Weighted Eigenvalues

**Proposition 2.1.**  $\rho_\ell = \lambda_\ell/d_\ell = \ell(\ell + 2)/(\ell + 1)^2 = 1 - (\ell + 1)^{-2}$ , so  $\rho_\ell \in (0, 1)$  strictly. The supremum of any ratio is  $\sup_{\ell > \ell'} \rho_\ell/\rho_{\ell'} < 1/\rho_1 = 4/3 \approx 1.333$ .

Representative values:  $\rho_1 = 3/4$ ,  $\rho_2 = 8/9$ ,  $\rho_{24} = 624/625$ . All density-weighted ratios are bounded by 4/3; target  $R^* \approx 206.77$  is structurally inaccessible. **CLOSED.**

## 3 Rule 2 — Heat Kernel at Natural Times

The heat trace on  $S^3$  and its small- $t$  asymptotic (unit sphere,  $\text{vol} = 2\pi^2$ , curvature coefficient  $a_1 = 1$ ):

$$K(t) = \sum_{\ell \geq 0} (\ell+1)^2 e^{-\lambda_\ell t}, \quad K(t) \sim \frac{\sqrt{\pi}}{4} t^{-3/2} (1 + t + O(t^2)), \quad t \rightarrow 0^+. \quad (2)$$

With electron/muon times  $t_e$  and  $t_\mu = r t_e$ , the leading-order ratio is

$$\frac{K(t_e)}{K(t_\mu)} \approx r^{3/2}, \quad r^{3/2} = R^* \implies r = (R^*)^{2/3} \approx 34.97. \quad (3)$$

**Proposition 3.1.**  $\lambda_5^{3/2} = 35\sqrt{35} \approx 207.063$ , gap  $(207.063 - 206.768)/206.768 = 0.143\%$ . No other  $S^3$  eigenvalue  $\lambda_\ell^{3/2}$  lies within 1% of  $R^*$  ( $\lambda_4^{3/2} \approx 117.6$ , gap 43%;  $\lambda_6^{3/2} \approx 332.6$ , gap 61%).

The identification hypothesis is  $t_\mu/t_e = \lambda_5 = 35$ .  $(R^*)^{2/3} \approx 34.97$ : the fifth eigenvalue is the unique integer-valued spectral value satisfying this.

*Remark 3.2* (Sub-leading correction). At  $t_e = 1/T_\bullet \approx 0.00232$  the first-order correction factor  $(1 + t_e)/(1 + 35t_e) \approx 0.927$  gives  $207.06 \times 0.927 \approx 191.9$  (gap  $\approx 7.2\%$ ). The leading-order near-miss is a ratio property that holds independently of the absolute scale  $t_e$ ; the sub-leading behaviour requires a geometric pinning of  $t_e$ .

**PROMISING.** Near-miss  $\lambda_5^{3/2} \approx 207.06$ , gap 0.14%.

## 4 Rule 3 — Spectral Zeta Ratios

$\zeta(s) = \sum_{\ell \geq 1} (\ell + 1)^2 [\ell(\ell + 2)]^{-s}$  converges iff  $s > 3/2$  (terms  $\sim \ell^{2-2s}$ ). Pole at  $s = 3/2$  with residue  $(2\sqrt{\pi})^{-1}$ . At  $s = 2$ : partial sum to  $\ell = 200$  gives  $\zeta(2) \approx 0.880$ ; all convergent values are  $O(1)$ . Residue ratios at the pole are trivially 1. Individual-term tuning to hit  $R^*$  introduces the same unjustified free parameter as Rule 2. **CLOSED.**

## 5 Rule 4 — GON Lapse

The GON lapse  $m = |\sin 2\beta|$  gives  $m_\mu/m_e = \sin 2\beta_\mu/\sin 2\beta_e$  only if  $\sin 2\beta_e \leq 1/R^* \approx 4.84 \times 10^{-3}$ . The smallest TOE-natural angle  $\pi/(\alpha^{-1})^2 \approx 1.67 \times 10^{-4}$  rad gives  $\sin(2\beta_e) \approx 3.35 \times 10^{-4}$ , requiring  $\beta_\mu \approx \pi/90.8$ ; the denominator  $90.8 \approx \alpha^{-1}/1.51$  has no canonical TOE derivation. For canonical angles  $\beta \in \{\pi/6, \pi/4, \pi/3\}$  the maximum lapse ratio is  $2/\sqrt{3} \approx 1.155$ , far below  $R^*$ . **CLOSED.**

## 6 Rule 5 — Moment Hierarchy

The  $n$ -th regulated moment  $\mu_n(t) = (-\partial_t)^n K(t)$  satisfies  $\mu_1(t)/\mu_0(t) \sim 3/(2t)$  as  $t \rightarrow 0^+$ . Ratios of  $\mu_0$  at two times reduce to  $(t_\mu/t_e)^{3/2}$  — exactly Rule 2. No new spectral information is accessed. **CLOSED (reduces to §3).**

## 7 Rule 6 — Windowed Spectrum

Window  $W = [\omega_0, T_\bullet] \approx [7.752, 430.5]$  contains levels  $\ell = 2, \dots, 19$  ( $\lambda_2 = 8 > \omega_0$ ;  $\lambda_{19} = 399 \leq T_\bullet$ ;  $\lambda_{20} = 440 > T_\bullet$ ;  $\lambda_1 = 3 < \omega_0$ ). Eighteen levels total. Multiplicity-weighted sum:  $S_W = \sum_{\ell=2}^{19} d_\ell \lambda_\ell = 719784$ . Ratio  $S_W/(d_1 \lambda_1) \approx 59982$ : three orders of magnitude above  $R^*$ . By the  $S^3$  Weyl law ( $N(\lambda) \sim \lambda^{3/2}$ ) window sum ratios grow polynomially and cannot produce  $O(200)$  values between adjacent windows. **CLOSED.**

## 8 Rule 7 — Casimir-Type Combinations

Searching for  $\alpha_1 \lambda_{\ell_1} + \alpha_2 \lambda_{\ell_2} = R^*(\alpha_1 \lambda_{\ell_3} + \alpha_2 \lambda_{\ell_4})$  with TOE-natural  $\alpha_i$ :

$$\lambda_{24}/\lambda_1 = 624/3 = 208.000, \quad \text{gap} = (208.000 - 206.768)/206.768 = 0.596\%. \quad (4)$$

$$\frac{3}{2}\alpha^{-1} = \frac{3}{2} \times 137.036 = 205.554, \quad \text{gap} = (206.768 - 205.554)/206.768 = 0.587\%. \quad (5)$$

Both near-misses are at  $\sim 0.6\%$ , inferior to the 0.14% of §3. The constant  $\omega_0 \approx 7.752$  is not a  $S^3$  eigenvalue, so  $\omega_0$ -weighted combinations produce no exact spectral expression. **CLOSED (near-misses at 0.6%).**

## 9 Summary Table

| Rule                 | Result    | Best value                     | Gap     |
|----------------------|-----------|--------------------------------|---------|
| §2 Density-weighted  | CLOSED    | $\sup \leq 4/3$                | $> 100$ |
| §3 Heat kernel       | PROMISING | $\lambda_5^{3/2} = 207.063$    | 0.143%  |
| §4 Spectral zeta     | CLOSED    | $\zeta$ -ratios $O(1)$         | $> 100$ |
| §5 GON lapse         | CLOSED    | No natural $\beta$ -pair       | $> 100$ |
| §6 Moment hierarchy  | CLOSED    | Reduces to §3                  | —       |
| §7 Windowed spectrum | CLOSED    | Scale mismatch $O(10^4)$       | $> 100$ |
| §8 Casimir combns.   | CLOSED    | $\lambda_{24}/\lambda_1 = 208$ | 0.596%  |

**Key numerical identity.**  $(R^*)^{2/3} = 206.768^{2/3} \approx 34.97 \approx \lambda_5 = 35$ ; gap 0.09%. By Proposition 3.1,  $\lambda_5$  is the unique  $S^3$  eigenvalue whose  $\frac{3}{2}$ -power matches  $R^*$  to better than 1%.

## 10 Conclusion

Six rules are definitively closed: density-weighted ratios are bounded by  $4/3$ , spectral-zeta global values are  $O(1)$ , GON lapse values require unphysical pole proximity, windowed sums have  $O(10^4)$  scale mismatch, moment hierarchy reduces to heat kernel, and Casimir near-misses fall at 0.6%.

**Open with single candidate.** The heat-kernel asymptotic scaling  $K(t_e)/K(t_\mu) \approx (t_\mu/t_e)^{3/2}$  matches  $R^*$  when  $t_\mu/t_e = \lambda_5 = 35$ , yielding  $35^{3/2} = 207.06$  (gap 0.14%). This is the closest approach found across all seven routes. The candidate requires a geometric argument for why the muon heat-kernel time exceeds the electron time by  $\lambda_5$ ; no such argument is currently available.

**Constraint on P18-Conj1.** Any valid Route-B identification rule must produce the factor  $\lambda_5^{3/2}$  or account for the 0.14% residual. The problem reduces to: *derive  $t_\mu/t_e = \lambda_5$  from the TOE geometry.*

*Next:* Addendum 195 investigates whether renormalisation-group running from the  $\hat{\mathcal{O}}$  scale to the physical scale accounts for the 0.14% residual gap.