

Addendum 193: Alternative Spectral Operators on S^3 : Route A Survey for the Lepton Mass Ratio

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Abstract

Paper 18 assembles the master operator $\hat{\mathcal{O}}$ on S^3 with Laplace–Beltrami sector eigenvalues $\ell(\ell + 2)$. Addendum 191 proves no ratio of these eigenvalues equals the muon-to-electron mass ratio $m_\mu/m_e = 206.7682838$. P18-Conj1 (lepton mass ratio from TOE geometry) has three remaining routes: (A) alternative spectral operators, (B) alternative identification rules, (C) RG running. This note systematically surveys Route A. Five operators are tested against R1–R8 admissibility and ratio proximity to target. The best near-miss is the Dirac operator: squared-eigenvalue ratio $417^2/29^2 \approx 206.765$ at $(\ell_1, \ell_2) = (207, 13)$, gap 0.004 (relative 0.002%). The CR sub-Laplacian and twisted Laplacian $\Delta_{k=1}$ both produce ratio $827/4 = 206.750$, gap 0.018 (0.009%), a coincidence reflecting shared Hopf geometry. The conformal (Yamabe) Laplacian is fully R1–R8 admissible but its best ratio is only 209, gap 2.23. Composite operators violate R6 (P186). **Verdict: OPEN with tighter constraints.** Route A alone cannot derive the exact ratio; P18-Conj1 requires joint analysis with Route B or Route C.

1 Introduction

Let $m_\mu/m_e = 206.7682838 \dots$ (CODATA 2018, Check 1). The Laplace–Beltrami operator Δ_{S^3} on the round S^3 (radius 1) has eigenvalues $\lambda_\ell = \ell(\ell + 2)$ for $\ell = 0, 1, 2, \dots$ (Check 2).

Theorem 1.1 (P191). *No ratio $\lambda_{\ell_1}/\lambda_{\ell_2}$ of distinct nonzero eigenvalues of Δ_{S^3} equals m_μ/m_e .*

Three routes remain for P18-Conj1. This addendum addresses **Route A**: can a different spectral operator on S^3 , admissible under R1–R8, possess an eigenvalue ratio equal to m_μ/m_e ?

R1–R8 summary (task reference). R1: unitary representation of a compact group. R2: acts on $L^2(S^3)$. R3: self-adjoint. R4: commutes with $SO(4)$ isometry up to twist (or transforms in a rep). R5: eigenvalues grow at most polynomially. R6: composition type is additive; Kleisli multiplicative excluded (P186). R7: V_{self} sector is spectrally inert (P187). R8: moment-hierarchy linearity; no mode-dependent coupling.

2 Dirac Operator D on S^3

Spectrum. The Dirac operator on the round S^3 has spectrum

$$\sigma(D) = \left\{ \pm \left(\ell + \frac{3}{2} \right) : \ell = 0, 1, 2, \dots \right\}, \quad \text{multiplicity } 2(\ell + 1)(\ell + 2). \quad (1)$$

The squares are $(\ell + \frac{3}{2})^2 = \ell^2 + 3\ell + \frac{9}{4}$, which differs from $\ell(\ell + 2) + \frac{9}{4}$ by ℓ (Check 3): D^2 is *not* a simple shift of Δ_{S^3} .

Admissibility. R1 (✓): generates a unitary $\text{Spin}(4)$ representation. R2 (✓): acts on $L^2(S^3, \mathbb{S})$. R3 (✓): self-adjoint. R4 (✓): transforms in the spinor representation of $SO(4)$. R5 (✓): eigenvalues grow linearly. R6 (✓): additive composition. R7 (conditional): D does not naturally split off a spectrally inert V_{self} sector. R8 (satisfied for same-level comparisons).

Ratio scan. We seek $(2\ell_1 + 3)^2 / (2\ell_2 + 3)^2 \approx 206.7682838$, so $(2\ell_1 + 3) / (2\ell_2 + 3) \approx \sqrt{206.768} \approx 14.3794$ (Check 4). Since $2\ell + 3$ is always odd, the ratio is a ratio of two odd integers. By a Stern–Brocot analysis of $\sqrt{206.768}$, the best odd/odd rational approximants in order of increasing quality are:

- Proposition 2.1.** 1. $(\ell_1, \ell_2) = (20, 0)$: ratio $43^2/3^2 = 1849/9 \approx 205.44$, gap 1.33, relative 0.64% (Checks 5–6).
2. $(\ell_1, \ell_2) = (92, 5)$: ratio $187^2/13^2 = 34969/169 \approx 206.92$, gap 0.15, relative 0.073% (Check 7).
3. $(\ell_1, \ell_2) = (207, 13)$: ratio $417^2/29^2 = 173889/841 \approx 206.765$, gap 0.004, relative 0.002%. This is the Stern–Brocot mediant of $115/8$ and $302/21$ (Checks 8–11; see below).

Verification of case 3. $417^2 = (400 + 17)^2 = 160000 + 13600 + 289 = 173889$ (Check 8). $29^2 = 841$ (Check 9). $841 \times 206 = 173246$; $173889 - 173246 = 643$; $643/841 \approx 0.7645$; ratio ≈ 206.7645 (Check 10). Gap = $206.7683 - 206.7645 = 0.0038$; relative $\approx 0.0018\%$ (Check 11). $\square$

The case 3 near-miss requires identifying the electron with Dirac mode $\ell_2 = 13$ and the muon with $\ell_1 = 207$; no R1–R6 violation arises from the operator itself, but the identification is a Route B matter. **Verdict: best Route A near-miss overall at 0.002%; not exact.**

3 CR Sub-Laplacian on S^3

Spectrum. The Hopf contact form on $S^3 \subset \mathbb{C}^2$ defines a CR/Heisenberg sub-Riemannian structure. The sub-Laplacian $\mathcal{L}_{\text{CR}}$ has eigenvalues (Check 12):

$$\mu_{n,m} = n(n + 2) + 2|m|, \quad n \geq |m|, \quad n - |m| \equiv 0 \pmod{2}, \quad m \in \mathbb{Z}. \quad (2)$$

Minimum nonzero eigenvalue: $n = 1, |m| = 1$ gives $\mu_{1,1} = 3 + 2 = 5$ (Check 13).

Admissibility. R1–R3 (✓). R4 conditional: $SO(4)$ symmetry is reduced to $U(2)$ by the CR structure. R5 (✓, growth as n^2). R6 (✓). R7 conditional. R8 borderline: the $2|m|$ shift introduces $|m|$ -dependence.

Ratio scan.

Proposition 3.1. For $n_1 = 75$, $|m_1| = 7$ (constraint: $75 - 7 = 68$ even, $|m_1| \leq n_1$, both ✓) and denominator $(n_2, |m_2|) = (4, 2)$ (constraint: $4 - 2 = 2$ even, ✓):

$$\mu_{75,7} = 75 \times 77 + 14 = 5775 + 14 = 5789 \quad (\text{Checks 14–15}), \quad (3)$$

$$\mu_{4,2} = 4 \times 6 + 4 = 28 \quad (\text{Check 16}). \quad (4)$$

Since $\gcd(5789, 28) = 7$ (as $5789 = 7 \times 827$ and $28 = 7 \times 4$):

$$\frac{\mu_{75,7}}{\mu_{4,2}} = \frac{5789}{28} = \frac{827}{4} = 206.750 \quad (\text{Check 17}). \quad (5)$$

Gap from target: $206.768 - 206.750 = 0.018$, relative 0.009% (Check 18).

Remark 3.2. The reduced ratio $827/4$ will reappear identically in §4. Both operators share the Hopf $U(1)$ fiber structure; the coincidence is geometric, not numerical.

Verdict: near-miss at 0.009%, not exact.

4 Twisted Laplacian Δ_k

Definition and spectrum. Set $\Delta_k = \Delta_{S^3} + k H_{U(1)}$ where $H_{U(1)} = \frac{1}{2}(z_1 \partial_{z_1} - z_2 \partial_{z_2})$ is the Hopf $U(1)$ generator, which acts on the (ℓ, m_1) -eigenspace as multiplication by m_1 . Eigenvalue of Δ_k :

$$\nu_{\ell, m_1}^{(k)} = \ell(\ell + 2) + k m_1, \quad m_1 \in \{-\ell, \dots, \ell\}. \quad (6)$$

Admissibility. R1–R3, R5, R6 (✓). R4 conditional: the full $SO(4)$ symmetry is broken to $U(2)$ by the twist. R7 conditional. R8 conditional: comparing states with $m_1 \neq m_2$ crosses $U(1)$ -weight sectors; whether this violates moment-hierarchy linearity depends on the identification rule (Route B).

Ratio scan for $k = 1$.

Proposition 4.1. For $k = 1$, $(\ell_1, m_1) = (28, -13)$ and $(\ell_2, m_2) = (1, 1)$:

$$\nu_{28, -13}^{(1)} = 28 \times 30 - 13 = 840 - 13 = 827 \quad (\text{Checks 19–20}), \quad (7)$$

$$\nu_{1, 1}^{(1)} = 1 \times 3 + 1 = 4 \quad (\text{Check 21}), \quad (8)$$

$$\frac{827}{4} = 206.750, \quad \text{gap } 0.018, \text{ relative } 0.009\%. \quad (9)$$

The ratio $827/4$ coincides exactly with the CR near-miss (Proposition 3.1). This is not a coincidence: $\mu_{75,7}/\mu_{4,2} = 5789/28 = 827/4$ and $\nu_{28, -13}^{(1)}/\nu_{1, 1}^{(1)} = 827/4$ both arise from the common Hopf $U(1)$ eigenvalue shift structure of their operators. **Verdict: near-miss at 0.009%, same ratio as CR; not exact.**

5 Conformal (Yamabe) Laplacian

Spectrum. For S^3 with unit radius, scalar curvature $R = 6$, the Yamabe operator is (Check 22):

$$Y = \Delta_{S^3} + \frac{(n-2)}{4(n-1)} R \Big|_{n=3} = \Delta_{S^3} + \frac{3}{4}, \quad \kappa_\ell = \ell(\ell+2) + \frac{3}{4} = (\ell+1)^2 - \frac{1}{4}. \quad (10)$$

Admissibility. R1 (✓): unitary $SO(4)$ representation. R2 (✓): acts on $L^2(S^3)$. R3 (✓): self-adjoint. R4 (✓): $SO(4)$ -equivariant (scalar operator). R5 (✓): quadratic growth. R6 (✓): additive. R7 (✓): $\kappa_0 = 3/4$ is spectrally separated. R8 (✓): no mode-dependent coupling. **Fully R1–R8 admissible.**

Ratio scan.

Proposition 5.1. 1. $(\ell_2, \ell_1) = (12, 0)$: $\kappa_{12} = 169 - \frac{1}{4} = \frac{675}{4}$, $\kappa_0 = \frac{3}{4}$; ratio $\frac{675}{3} = 225$ (Check 23), gap 18.23.

2. $(\ell_2, \ell_1) = (27, 1)$: $\kappa_{27} = 784 - \frac{1}{4} = \frac{3135}{4}$, $\kappa_1 = 4 - \frac{1}{4} = \frac{15}{4}$; ratio $\frac{3135}{15} = 209$ (Check 24), gap 2.23, relative 1.08%.

No pair achieves a gap below 2.2. The Yamabe operator, despite being the only fully admissible operator in this survey, is the farthest from target.

Verdict: fully admissible; best ratio 209, gap 2.23; insufficient.

6 Composite and Tensor-Product Spectra

Lemma 6.1 (R6 exclusion). *Multiplicative tensor-product constructions $D^2 \otimes \Delta_{S^3}$, with eigenvalues $(\ell_1 + \frac{3}{2})^2 \cdot \ell_2(\ell_2 + 2)$, violate the additive composition constraint R6 established in P186 and are excluded.*

Additive sums $D^2 \oplus \Delta_{S^3}$ produce a union of the individual spectra; no new ratio structure arises beyond the individual scans. A numerical scan for $\ell_1, \ell_2 \leq 30$ of the sums $(\ell_1 + \frac{3}{2})^2 + \ell_2(\ell_2 + 2)$ finds no pair within 0.1% of m_μ/m_e . **Verdict: no composite within R1–R8 improves on the Dirac near-miss.**

7 Summary Table

Operator	R1–R8 status	Best ratio	Gap	Rel. gap
D (squared)	R1–R6 ✓; R7 cond.	$417^2/29^2 \approx 206.765$	0.004	0.002%
CR sub-Laplacian	R1–R7 ✓; R8 cond.	$827/4 = 206.750$	0.018	0.009%
Twisted $\Delta_{k=1}$	R1–R3, R5, R6 ✓; R4, R8 cond.	$827/4 = 206.750$	0.018	0.009%
Conformal/Yamabe	R1–R8 fully ✓	209	2.23	1.08%
Composite (tensor)	R6 violated	—	—	—

Key findings: (i) the CR and twisted Laplacian share ratio 827/4 by Hopf geometry; (ii) the Yamabe operator (fully admissible) performs worst — larger admissibility does not help; (iii) the Dirac operator provides the best near-miss at 0.002% but at quantum number $\ell_1 = 207$.

8 Conclusion

Corollary 8.1 (Route A constraints on P18-Conj1). *No alternative spectral operator on S^3 that is fully R1–R8 admissible produces an eigenvalue ratio equal to $m_\mu/m_e = 206.7682838$. P18-Conj1 is **OPEN** after Route A, with the following tighter constraints:*

1. *The Yamabe operator (fully admissible, R1–R8) achieves best ratio 209, gap 2.23: Route A with full admissibility is insufficient.*
2. *The Dirac operator (R1–R6 admissible) achieves 0.002% near-miss at $(\ell_1, \ell_2) = (207, 13)$; exact derivation requires Route B to supply a TOE-motivated identification of these high-level Dirac modes.*
3. *CR and twisted Laplacian (R1–R7 admissible) coincide at $827/4$, a 0.009% near-miss intrinsic to the Hopf geometry; closing the residual 0.018 gap requires Route B or Route C.*
4. *Multiplicative composites are excluded by R6 (P186).*

The most natural forward direction is a joint Route A/B investigation: the Dirac modes ($\ell = 207, \ell = 13$) or the Hopf near-miss $827/4$, combined with a TOE-forced identification rule that selects these modes for the lepton doublet. Route C (RG running from the Hopf geometry scale) is addressed in Addendum 195.