

Non-perturbative $H_{U(1)}^\perp$ and the Lepton Mass
Formula:
Resolution of Open Problem 7.1 from
Addendum P188

Addendum 190 to the TOE Corpus

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Abstract

Addendum P188 characterised $H_{U(1)}^\perp = \partial/\partial\phi \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$ — the Killing-orthogonal complement of $H_{U(1)}$ in the two-dimensional F_4 -centralising subalgebra — and showed that adding $\varepsilon \cdot H_{U(1)}^\perp$ as a 9th sector of $\hat{O}$ in first-order perturbation theory requires a coupling $\kappa \approx 728$ (physical convention: $|c| \approx 1.72$) to close the lepton mass formula gap ($m_\mu/m_e: 52.8 \rightarrow 206.8$). P188 Open Problem 7.1 asked whether any *non-perturbative* inclusion of $H_{U(1)}^\perp$ — via $(H_{U(1)}^\perp)^2$, higher polynomials $f(H_{U(1)}^\perp)$, or operadic compositions — can do better.

This addendum proves three structural results that reduce Open Problem 7.1.

Theorem 3.1: The second-order perturbation correction from $\varepsilon \cdot H_{U(1)}^\perp$ vanishes identically. Since $H_{U(1)}^\perp = \partial/\partial\phi$ is already diagonal in the Wigner D -function basis — the same basis that diagonalises Δ_{S^3} — all off-diagonal matrix elements $\langle m | H_{U(1)}^\perp | n \rangle$ with $m \neq n$ are zero, and the standard second-order sum is empty.

Theorem 4.1: For *any* function f applied to $H_{U(1)}^\perp$, the operator $f(H_{U(1)}^\perp)$ remains diagonal in the D -function basis with eigenvalue $f(im_1)$ on D_{m_1, m_2}^ℓ . For any electron–muon identification in which both states carry the same azimuthal quantum number m_1 , the correction $f(im_1)$ cancels exactly in the spectral gap $\Delta E = E_\mu - E_e$, leaving the mass ratio unchanged regardless of the coefficient. This is the same spectral-inertness mechanism identified for V_{self} in P187.

Theorem 5.1: For cross- m_1 identifications (electron and muon at different m_1 values), the correction is $c(m_{1,e}^2 - m_{1,\mu}^2)$ from $c \cdot (H_{U(1)}^\perp)^2$. For the natural identification (electron at $(\ell=1, m_1=1)$, muon at $(\ell=2, m_1=0)$), the required $c \approx 1.720$ is *naturally sized* — within 0.7% of $\sqrt{3}$ — but no physical principle within $R1$ – $R8$ or the TOE geometry selects a specific m_1 for either lepton. The case is therefore *structurally possible but not predictive*.

The operadic composition replacing $\zeta H_{U(1)}$ with $\zeta(H_{U(1)} + f \cdot H_{U(1)}^\perp)$ for non-constant $f = f(\chi)$ introduces ℓ -mixing and is genuinely open; the smooth candidate $f(\chi) = \sin \chi$ is identified as the natural next target.

Verdict: OPEN — the non-perturbative $H_{U(1)}^\perp$ question is structurally reduced but not closed.

Numerical companion: `verify_P190.py` (83/83 checks pass, `mp.dps = 55`).

1 Background and Open Problem 7.1

1.1 The P188 result recalled

The two-dimensional F_4 -centralising Cartan complement $\mathfrak{z} \subset \mathfrak{h}_{E_6}$ (P176, P184) has an orthonormal basis

$$H_{U(1)} = \frac{1}{2}(z_1 \partial_{z_1} + z_2 \partial_{z_2}) \quad \text{and} \quad H_{U(1)}^\perp = \frac{1}{2}(z_1 \partial_{z_1} - z_2 \partial_{z_2}), \quad (1)$$

with $K(H_{U(1)}, H_{U(1)}) = K(H_{U(1)}^\perp, H_{U(1)}^\perp) = 24$ and $K(H_{U(1)}, H_{U(1)}^\perp) = 0$ (P188 Theorem 3.1). On S^3 in Hopf coordinates (χ, ϕ, ψ) : $H_{U(1)} = \partial/\partial\psi$ (Hopf fibre generator, nowhere-vanishing) and $H_{U(1)}^\perp = \partial/\partial\phi$ (azimuthal base generator, vanishing at the Hopf poles $\chi = 0, \pi$; P188 Theorem 4.1).

P188 Theorem 6.1 showed that adding $\varepsilon \cdot H_{U(1)}^\perp$ as a 9th sector in first-order perturbation theory splits the ℓ -th eigenspace of Δ_{S^3} by azimuthal quantum number m_1 :

$$E_{\ell, m_1, m_2}^{(1)} = \ell(\ell + 2) + \kappa m_1, \quad \kappa \in \mathbb{R} \quad (\text{with } \varepsilon = -i\kappa). \quad (2)$$

Closing the mass-formula gap (P187 convention: $m_\mu/m_e: 52.8 \rightarrow 206.8$ using $\mu_1/\mu_0 \approx 0.7933$) requires $\kappa \approx 1.72$ for the identified P187 transitions and $\kappa \approx 728$ in the P18 eigenvalue convention ($\mu_1/\mu_0 = \alpha$). P188 declared the latter implausible at any natural coupling. Both estimates agree that first-order perturbation theory alone cannot close the gap with a naturally-normalised coupling.

1.2 Open Problem 7.1 from P188

P188 Open Problem 7.1(iii) asks: does $(H_{U(1)}^\perp)^2 = (\partial/\partial\phi)^2$ as a second-order operator produce a spectrum with a ratio closer to 207? Parts (i) and (ii) ask more broadly whether any assembly incorporating $H_{U(1)}^\perp$ — additively or via operadic/Kleisli composition — is admissible and improves the mass formula.

1.3 Key tool: Wigner D -functions on S^3

The Wigner D -functions $D_{m_1, m_2}^\ell(\phi, \chi, \psi)$ form an orthonormal basis of $L^2(S^3)$ (Gel'fand–Tsetlin basis for $SU(2)$). In Hopf coordinates:

$$D_{m_1, m_2}^\ell(\phi, \chi, \psi) = e^{im_1\phi} d_{m_1, m_2}^\ell(\chi) e^{im_2\psi}, \quad (3)$$

where $d_{m_1, m_2}^\ell(\chi)$ are the Wigner small- d matrices and $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, $m_1, m_2 \in \{-\ell, \dots, \ell\}$. The D -functions are simultaneous eigenfunctions:

$$\Delta_{S^3} D_{m_1, m_2}^\ell = \ell(\ell+2) D_{m_1, m_2}^\ell, \quad \frac{\partial}{\partial\psi} D_{m_1, m_2}^\ell = im_2 D_{m_1, m_2}^\ell, \quad \frac{\partial}{\partial\phi} D_{m_1, m_2}^\ell = im_1 D_{m_1, m_2}^\ell. \quad (4)$$

The third identity follows directly from (3): differentiating $e^{im_1\phi}$ with respect to ϕ gives $im_1 e^{im_1\phi}$, while $d_{m_1, m_2}^\ell(\chi)$ and $e^{im_2\psi}$ are unaffected. This means $H_{U(1)}^\perp = \partial/\partial\phi$ is *already diagonal* in the D -function basis.

2 The P18 Mass Formula and the Spectral Gap

The P18 mass formula (Conjecture 5.1) is:

$$\frac{m_n}{m_e} = \exp\left(\frac{\mu_1}{\mu_0} \Delta E_n\right), \quad (5)$$

where $\mu_k = \int_0^1 r^k \rho(r) dr$ are the radial moments of the geometric density $\rho(r) = 16\pi^3 r^3 + 3\pi^2 r^2 + 2\pi r$ (P03):

$$\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036 = \alpha^{-1}, \quad \mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} \approx 108.717, \quad \frac{\mu_1}{\mu_0} \approx 0.7933. \quad (6)$$

P187 (following P18 Remark 5.2) computes: with Δ_{S^3} alone, the electron–muon spectral gap is $\Delta E = \ell(\ell+2)|_{\ell=2} - \ell(\ell+2)|_{\ell=1} = 8 - 3 = 5$, giving:

$$\left. \frac{m_\mu}{m_e} \right|_{\text{unperturbed}} = \exp(0.7933 \times 5) = e^{3.967} \approx 52.8, \quad (7)$$

versus the experimental value $m_\mu/m_e \approx 206.77$. The required spectral gap to match observation is:

$$\Delta E^* = \frac{\mu_0}{\mu_1} \ln(206.77) \approx \frac{\ln(206.77)}{0.7933} \approx 6.720, \quad (8)$$

so the canonical assembly falls short by $\delta E = 6.720 - 5 = 1.720$.

Remark 2.1 (Conventions for μ_1/μ_0). P188 Theorem 6.1 uses $\mu_1/\mu_0 = \alpha \approx 1/137$ (the P18 eigenvalue convention, where the mass formula exponent is normalised to the fine-structure constant). Under this convention, the required eigenvalue at $\ell = 1$ is $3 + \kappa = (\mu_0/\mu_1) \ln(206.8) \approx 730.9$, giving $\kappa \approx 728$. The present paper uses the physical moment ratio $\mu_1/\mu_0 \approx 0.7933$ throughout, which gives the smaller shortfall $\delta E \approx 1.720$. Both conventions agree on the qualitative conclusion: first-order $\varepsilon \cdot H_{U(1)}^\perp$ requires a coupling of the same order as the target shift itself.

3 Second-Order Perturbation from $\varepsilon \cdot H_{U(1)}^\perp$

Theorem 3.1 (Second-order correction vanishes). *The second-order perturbation-theory correction to any eigenvalue of Δ_{S^3} from the perturbation $V = \varepsilon \cdot H_{U(1)}^\perp$ is identically zero for all $\varepsilon \in \mathbb{C}$.*

Proof. The second-order correction to the eigenvalue of state $|n\rangle = D_{m_1, m_2}^\ell$ is (degenerate second-order perturbation theory):

$$E_n^{(2)} = \sum_{|m\rangle \notin \mathcal{H}_\ell} \frac{|\langle m|V|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}, \quad (9)$$

where the sum is over states $|m\rangle = D_{m'_1, m'_2}^{\ell'}$ with $\ell' \neq \ell$ (outside the degenerate subspace $\mathcal{H}_\ell$ of Δ_{S^3} at eigenvalue $\ell(\ell+2)$).

From (4), $V = \varepsilon \cdot \partial/\partial\phi$ acts on D_{m_1, m_2}^ℓ as:

$$V D_{m_1, m_2}^\ell = \varepsilon i m_1 D_{m_1, m_2}^\ell.$$

Therefore:

$$\langle D_{m'_1, m'_2}^{\ell'} | V | D_{m_1, m_2}^\ell \rangle = \varepsilon i m_1 \langle D_{m'_1, m'_2}^{\ell'} | D_{m_1, m_2}^\ell \rangle = \varepsilon i m_1 \cdot \delta_{\ell'\ell} \delta_{m'_1 m_1} \delta_{m'_2 m_2}. \quad (10)$$

For $\ell' \neq \ell$, the delta $\delta_{\ell'\ell} = 0$ kills every off-diagonal matrix element in the sum (9). Hence $E_n^{(2)} = 0$.

The argument is independent of ε and of the specific state n : V maps every D -function to a scalar multiple of itself, leaving all other states orthogonal. No degenerate mixing within the same ℓ -eigenspace contributes to the inter-space sum (9) because the Δ_{S^3} degenerate subspaces at different ℓ are orthogonal. $\square$

Corollary 3.2 (Perturbation from $\varepsilon H_{U(1)}^\perp$ is exactly first-order). *The full perturbative expansion of the eigenvalue of $\Delta_{S^3} + \varepsilon H_{U(1)}^\perp$ terminates at first order. The exact eigenvalue on D_{m_1, m_2}^ℓ is:*

$$E_{\ell, m_1, m_2}^\varepsilon = \ell(\ell+2) + \varepsilon i m_1. \quad (11)$$

This is not a perturbative approximation: since $H_{U(1)}^\perp$ is diagonal in the D -function basis, the sum $\Delta_{S^3} + \varepsilon H_{U(1)}^\perp$ is simultaneously diagonalised by the same basis, and the eigenvalues are exact.

4 Non-perturbative Polynomial $f(H_{U(1)}^\perp)$

Theorem 4.1 (Any $f(H_{U(1)}^\perp)$ is spectrally inert for fixed- m_1 transitions). *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be any function with $f(im_1) \in \mathbb{R}$ for all integers m_1 (so that the modified eigenvalues are real). Define the modified operator $\hat{O}' = \hat{O} + c \cdot f(H_{U(1)}^\perp)$ for $c \in \mathbb{R}$. Then:*

(i) $f(H_{U(1)}^\perp)$ is diagonal in the D -function basis with eigenvalue $f(im_1)$:

$$f(H_{U(1)}^\perp) D_{m_1, m_2}^\ell = f(im_1) D_{m_1, m_2}^\ell. \quad (12)$$

(ii) The eigenvalue of $\hat{O}'$ on D_{m_1, m_2}^ℓ is:

$$E'_{\ell, m_1, m_2} = \ell(\ell + 2) + c f(im_1) \quad (+ \text{contributions from other sectors}). \quad (13)$$

(iii) For any electron state at (ℓ_e, m_1) and muon state at (ℓ_μ, m_1) with the same m_1 :

$$\Delta E' = E'_{\ell_\mu, m_1, \cdot} - E'_{\ell_e, m_1, \cdot} = \ell_\mu(\ell_\mu + 2) - \ell_e(\ell_e + 2), \quad (14)$$

independent of c and f . The mass ratio m_μ/m_e is unchanged.

Proof. (i). By the spectral mapping theorem: since $H_{U(1)}^\perp$ is diagonalised by the D -functions with eigenvalue im_1 , any function $f(H_{U(1)}^\perp)$ is diagonalised by the same basis with eigenvalue $f(im_1)$. This is exact (not perturbative); it follows from the fact that $H_{U(1)}^\perp$ is a normal operator on $L^2(S^3)$ and its spectral decomposition is the D -function expansion.

(ii). Direct from (i) and the linearity of the eigenvalue.

(iii). In (13), the correction $cf(im_1)$ is the same for the electron (state $D_{m_1, \cdot}^{\ell_e}$) and the muon (state $D_{m_1, \cdot}^{\ell_\mu}$) since both share the same m_1 . It cancels exactly in the difference (14). $\square$

Corollary 4.2 (P187 analogy: spectral inertness). *The mechanism of Theorem 4.1(iii) is structurally identical to the spectral inertness of V_{self} proved in P187 Corollary 7.3: in both cases, a perturbation that acts the same way on both lepton states cancels in the mass ratio. For V_{self} , the perturbation was radial (mode-uniform at leading order); for $f(H_{U(1)}^\perp)$, it is m_1 -uniform within any fixed- m_1 sector.*

Remark 4.3 ($(H_{U(1)}^\perp)^2$ spectrum explicitly). Taking $f(z) = -z^2$ (so $f(H_{U(1)}^\perp) = (\partial/\partial\phi)^2$):

$$(H_{U(1)}^\perp)^2 D_{m_1, m_2}^\ell = \left(\frac{\partial}{\partial\phi} \right)^2 D_{m_1, m_2}^\ell = (im_1)^2 D_{m_1, m_2}^\ell = -m_1^2 D_{m_1, m_2}^\ell.$$

The spectrum of $c \cdot (H_{U(1)}^\perp)^2$ on the ℓ -th eigenspace is $\{-cm_1^2 : m_1 = -\ell, \dots, \ell\}$. For $\ell = 1$: $\{0, -c\}$ (non-degenerate); for $\ell = 2$: $\{0, -c, -4c\}$.

For $c > 0$: the highest-energy states within each ℓ -sector are those with $m_1 = 0$; the lowest are $m_1 = \pm\ell$. For $c < 0$: the ordering is reversed. In either case, for the *same* m_1 at $\ell = 1$ and $\ell = 2$, the correction cancels and $\Delta E = 5$ is unchanged.

5 The Cross- m_1 Case

The fixed- m_1 cancellation (Theorem 4.1) forces us to ask whether a cross- m_1 identification — electron and muon at different m_1 values — can break the inertness.

Theorem 5.1 (Cross- m_1 gap and required coefficient). *For an operator $c \cdot (H_{U(1)}^\perp)^2$ added to $\hat{O}$, the spectral gap between an electron state at $(\ell_e, m_{1,e})$ and a muon state at $(\ell_\mu, m_{1,\mu})$ with $m_{1,e} \neq m_{1,\mu}$ is:*

$$\Delta E = \ell_\mu(\ell_\mu + 2) - \ell_e(\ell_e + 2) + c(m_{1,e}^2 - m_{1,\mu}^2). \quad (15)$$

Setting $\ell_e = 1$ and $\ell_\mu = 2$ (the P187 assignment):

$$\Delta E = 5 + c(m_{1,e}^2 - m_{1,\mu}^2). \quad (16)$$

The required $\Delta E^* \approx 6.720$ (from Section 2) is achieved for the coefficient:

$$c^* = \frac{\delta E}{m_{1,e}^2 - m_{1,\mu}^2}, \quad \delta E = \Delta E^* - 5 \approx 1.720, \quad (17)$$

provided $m_{1,e}^2 \neq m_{1,\mu}^2$.

(a) **Case A** ($m_{1,e} = 1, m_{1,\mu} = 0$): $c^* = 1.720/1 \approx 1.720$.

(b) **Case B** ($m_{1,e} = 0, m_{1,\mu} = 2$): $c^* = 1.720/(-4) \approx -0.430$.

(c) **Case C** ($m_{1,e} = 1, m_{1,\mu} = 2$): $c^* = 1.720/(-3) \approx -0.573$.

Proof. Direct substitution into (13) with $f(im_1) = -m_1^2$: $E'_{\ell,m_1} = \ell(\ell + 2) - cm_1^2$. Then $\Delta E = E'_{\ell_\mu, m_{1,\mu}} - E'_{\ell_e, m_{1,e}} = (\ell_\mu(\ell_\mu + 2) - cm_{1,\mu}^2) - (\ell_e(\ell_e + 2) - cm_{1,e}^2) = 5 + c(m_{1,e}^2 - m_{1,\mu}^2)$. $\square$

Remark 5.2 (Case A coefficient and $\sqrt{3}$). Case A gives $c^* \approx 1.720$. Numerically $\sqrt{3} \approx 1.732$, and $|c^* - \sqrt{3}|/\sqrt{3} \approx 0.67\%$. The value $\sqrt{3}$ appears naturally in the TOE root system geometry (e.g., the long-root norm ratio in $A_2 \subset G_2 \subset F_4$) and as the length of the non-zero weights of the adjoint representation of G_2 . This numerical proximity is noted as a potentially significant coincidence but is not proved here: no derivation of $c = \sqrt{3}$ from the TOE algebra currently exists.

Verification: $\exp(0.7933 \times (5 + 1.720)) = \exp(0.7933 \times 6.720) \approx 206.8$ (see `verify_P190.py` Check 8).

Remark 5.3 (Physical interpretation of cross- m_1). The identification of the physical electron with a specific m_1 value is not motivated by any principle within R1–R8 or the current TOE. The quantum number m_1 is the eigenvalue of $\partial/\partial\phi = H_{U(1)}^\perp$ (the azimuthal base rotation); in the canonical assembly $H_{U(1)}^\perp$ does not appear as an operator at all, so m_1 is unobservable and all states with the same ℓ but different m_1 are physically degenerate. Introducing $c \cdot (H_{U(1)}^\perp)^2$ breaks this degeneracy, but without a further principle specifying which m_1 -eigenstate corresponds to which lepton, the cross- m_1 identification is ambiguous. The three cases A, B, C in Theorem 5.1 produce the correct mass ratio for different and mutually exclusive state identifications.

This contrasts with the $\mathbb{Z}_3$ family symmetry (R6), which does provide a selection rule for the three-fold degeneracy in m_2 (the Hopf fibre quantum number): the T -eigenvalues $e^{im_2 \cdot 4\pi/3}$ label the three lepton families. No analogue of R6 exists for m_1 .

6 Operadic Composition with $H_{U(1)}^\perp$

P188 Open Problem 7.1(ii) also raises operadic (non-additive) composition. The canonical Perturb sector is $\text{Perturb} \leftrightarrow \zeta H_{U(1)}$ (P184 Theorem 5.3). We consider replacing this with $\zeta(H_{U(1)} + f \cdot H_{U(1)}^\perp)$ for a function f on S^3 .

6.1 Constant f : remains diagonal and inert

If f is a constant, the combined generator $\partial/\partial\psi + f \cdot \partial/\partial\phi$ has eigenvalue $i(m_2 + fm_1)$ on D_{m_1, m_2}^ℓ . This is again diagonal; for the Perturb sector, the eigenvalue contribution is $\zeta \cdot i(m_2 + fm_1)$. The dominant mass-ratio correction is still from Δ_{S^3} , which is unaffected. Constant f does not help.

6.2 Non-constant $f = f(\chi)$: ℓ -mixing, genuinely open

For $f = f(\chi)$ a smooth non-constant function of the polar angle χ on S^2 (the Hopf base), the operator $f(\chi) \cdot \partial/\partial\phi$ is *not* diagonal in the D -function basis. The matrix element

$$\langle D_{m'_1, m'_2}^{\ell'} | f(\chi) \cdot \partial/\partial\phi | D_{m_1, m_2}^\ell \rangle = im_1 \cdot \delta_{m'_1 m_1} \delta_{m'_2 m_2} \cdot \int_0^\pi f(\chi) d_{m_1, m_2}^{\ell'}(\chi) d_{m_1, m_2}^\ell(\chi) \sin \chi d\chi \quad (18)$$

connects eigenspaces at different ℓ' and ℓ (the χ -integral is generically non-zero for $\ell' \neq \ell$ when f is non-constant), creating ℓ -mixing and a qualitatively different spectrum.

Observation 6.1 (The smooth candidate $f(\chi) = \sin \chi$). The operator $\sin \chi \cdot \partial/\partial\phi$ is globally smooth on S^3 : the factor $\sin \chi \rightarrow 0$ as $\chi \rightarrow 0, \pi$ cancels the zeros of $\partial/\partial\phi$ at the Hopf poles. Contrast with $H_{U(1)}^\perp = \partial/\partial\phi$ alone, which has isolated zeros. If the additional smoothness requirement “every generator in $\hat{O}$ that acts on S^3 must be a globally smooth (nowhere-vanishing) vector field” were adopted as a geometric principle (P188 Remark 6.3), it would exclude $H_{U(1)}^\perp$ alone but admit $\sin \chi \cdot \partial/\partial\phi$ only if the smoothness criterion is satisfied by the *operator* rather than the underlying vector field.

In the sense of operators on $L^2(S^3)$: $\sin \chi \cdot \partial/\partial\phi$ is a well-defined bounded operator (the multiplication by $\sin \chi$ is bounded, and $\partial/\partial\phi$ is only unbounded due to the gradient-type structure). The full operator $\Delta_{S^3} + \zeta \sin \chi \cdot \partial/\partial\phi$ is not diagonal in the D -function basis, and its spectrum is not known in closed form. The ℓ -mixing matrix element (18) for $f = \sin \chi$ is:

$$\int_0^\pi \sin \chi \cdot d_{m_1, m_2}^{\ell'}(\chi) d_{m_1, m_2}^\ell(\chi) \sin \chi d\chi = \int_0^\pi d_{m_1, m_2}^{\ell'}(\chi) d_{m_1, m_2}^\ell(\chi) \sin^2 \chi d\chi. \quad (19)$$

This connects $\ell' = \ell$ and $\ell' = \ell \pm 1$ (by the Clebsch–Gordan rule for multiplication by $\sin \chi \sim Y_1^0$), and is non-zero for adjacent levels. The full spectral analysis of $\Delta_{S^3} + \zeta \sin \chi \cdot \partial/\partial\phi$ is the primary open direction identified by this addendum.

7 Admissibility under R1–R8

We check the three cases of $(H_{U(1)}^\perp)^2$ and $f(\chi) \cdot H_{U(1)}^\perp$ against R1–R8.

Proposition 7.1 ($(H_{U(1)}^\perp)^2$ is admissible under R1–R8). *Adding $c \cdot (H_{U(1)}^\perp)^2$ to $\hat{O}$ is not excluded by any of R1–R8:*

- *R2 (boundary dynamics on S^3): $(H_{U(1)}^\perp)^2$ acts on S^3 . No conflict.*
- *R6 ($\mathbb{Z}_3$ family symmetry): $[(\partial/\partial\phi)^2, T] = 0$ because T acts as $\psi \mapsto \psi + 4\pi/3$ (pure ψ -shift) and $(\partial/\partial\phi)^2$ acts on ϕ only. Verified for all (D_{m_1, m_2}^ℓ) in `verify_P190.py` Check 11.*
- *R1, R3–R5, R7, R8: identical analysis to P188 Theorem 5.1 for $H_{U(1)}^\perp$; $(H_{U(1)}^\perp)^2$ acts on the same angular sector as $H_{U(1)}^\perp$ and introduces no new bulk, fibre, or radial structure.*

The global smoothness criterion (P188 Remark 6.3): as a degree-2 polynomial in $\partial/\partial\phi$, the operator $(H_{U(1)}^\perp)^2$ has a different character than the vector field $H_{U(1)}^\perp$. As an operator on $L^2(S^3)$, $(\partial/\partial\phi)^2$ is a well-defined self-adjoint operator (the negative azimuthal Laplacian). It does not have “zeros” in the vector-field sense; the criterion of P188 Remark 6.3 does not apply to second-order operators.

Remark 7.2 (Missing physical constraint for $(H_{U(1)}^\perp)^2$). The absence of an R1–R8 veto does not establish that $(H_{U(1)}^\perp)^2$ is *required* or even *natural* in the assembly. The canonical assembly uses only operators derived from explicit TOE sector requirements; P184 derives $\text{Perturb} \leftrightarrow \zeta H_{U(1)}$ via the boundary restriction of the renormalization generator $\mathcal{R}$ (Euler decomposition + Hopf boundary limit), not by an $\mathfrak{z}$ -completeness principle. There is no analogous derivation of $(H_{U(1)}^\perp)^2$ from any TOE sector.

8 Verdict and Open Problem 7.1 Status

Table 1: Status of non-perturbative $H_{U(1)}^\perp$ cases

Case	Spectral effect	Status
$\varepsilon \cdot H_{U(1)}^\perp$ (1st order)	eigenvalue shift κm_1 ; $\kappa \approx 1.72$ needed	IMPLAUSIBLE (P188)
$\varepsilon \cdot H_{U(1)}^\perp$ (2nd order)	identically zero	RULED OUT
$c \cdot f(H_{U(1)}^\perp)$, fixed m_1	cancels in ΔE for any f, c	RULED OUT
$c \cdot (H_{U(1)}^\perp)^2$, cross- m_1	$\Delta E = 5 + c(m_1^2 - m_1'^2)$; $c^* \approx 1.72$	NO SELECTION RULE
$\zeta(H_{U(1)} + f_0 H_{U(1)}^\perp)$, const f_0	diagonal, inert	RULED OUT
$\zeta(H_{U(1)} + f(\chi) H_{U(1)}^\perp)$	ℓ -mixing, unknown spectrum	OPEN

Open Problem 8.1 (Open Problem 7.1 — reduced form). The non-perturbative $H_{U(1)}^\perp$ question reduces to the following after Theorems 3.1, 4.1, and 5.1:

- **Cross- m_1 selection rule.** Is there a physical or geometric principle — derivable from the TOE or from an extended set of admissibility requirements beyond R1–R8 — that selects specific m_1 quantum numbers for the electron and muon? If the answer is yes and the selected pair is among those in Theorem 5.1, then $c \cdot (H_{U(1)}^\perp)^2$ with a naturally-sized coefficient solves the mass formula problem. The near-miss $c^* \approx 1.720 \approx \sqrt{3}$ (to 0.7%) is suggestive; the derivation is open.

- (ii) **Smooth operadic composition.** What is the spectrum of $\Delta_{S^3} + \zeta \sin \chi \cdot \partial/\partial\phi$? Does it produce a first excited level with $\Delta E \approx 6.72$? Is the assembly admissible under R1–R8? The ℓ -mixing induced by $\sin \chi \cdot \partial/\partial\phi$ couples ℓ and $\ell \pm 1$ eigenspaces and could qualitatively restructure the spectrum; a numerical study or a Mathieu-equation reduction (the χ -equation may reduce to a form of Hill’s equation) is required.
- (iii) **Coefficient derivation.** If a cross- m_1 selection rule or smooth operadic composition produces a mass ratio close to 207 for some coefficient c , is that coefficient derivable from the TOE (e.g., from the Killing form, from the root geometry of $G_2 \subset F_4 \subset E_6$, or from the layer fractions)?

9 Conclusion

The main results of this addendum resolve the tractable parts of P188 Open Problem 7.1 and identify the remaining open direction:

$$\begin{array}{l}
 \varepsilon \cdot H_{U(1)}^\perp \text{ (2nd-order)} : \quad E_n^{(2)} = 0 \text{ exactly} \\
 c \cdot f(H_{U(1)}^\perp) \text{ (fixed-} m_1 \text{)} : \quad \Delta E = 5 \text{ for all } f, c \\
 c \cdot (H_{U(1)}^\perp)^2 \text{ (cross-} m_1 \text{, Case A)} : \quad c^* \approx 1.720 \approx \sqrt{3} \text{ (no selection rule)} \\
 \zeta(H_{U(1)} + f(\chi)H_{U(1)}^\perp) : \quad \text{OPEN} \text{ — } \ell\text{-mixing, spectrum unknown}
 \end{array} \tag{20}$$

The spectral-inertness theorem (Theorem 4.1) is the central structural result: it shows that the entire polynomial algebra generated by $H_{U(1)}^\perp$ is inert for mass ratios under fixed- m_1 identification, by exactly the same mechanism (cancellation in ΔE) that makes V_{self} inert (P187). This places $H_{U(1)}^\perp$ in the same category as V_{self} : structurally present, admissible under R1–R8, but unable to affect the lepton mass ratio through any additive non-perturbative mechanism.

The residual open case — operadic composition with $f(\chi) \cdot H_{U(1)}^\perp$ — is qualitatively different. It introduces ℓ -mixing (connecting the electron and muon sectors directly) and is not captured by the diagonal-operator analysis of this addendum. The smooth candidate $\sin \chi \cdot \partial/\partial\phi$ is globally smooth on S^3 , admissible under R1–R8, and geometrically natural; its spectral analysis is the primary open direction.

Status of Open Problem 7.1: partially resolved. The second-order and polynomial-diagonal sub-cases are closed (ruled out or spectrally inert). The cross- m_1 sub-case finds a naturally-sized coefficient but lacks a selection rule. The smooth operadic composition sub-case is open and constitutes the primary remaining research direction.

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