

The Orthogonal Direction $H_{U(1)}^\perp$ in the Centraliser $\mathfrak{z} \subset \mathfrak{h}_{E_6}$: Algebraic Characterisation, Admissibility under R1–R8, and Implications for the Mass Formula

Addendum 188 to the TOE Corpus

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Abstract

Addendum P184 (OP-H Formal Derivation) proved that the renormalization sector $\zeta\mathcal{R}$ of Paper 18’s master operator $\hat{O}$, restricted to the boundary S^3 , coincides with $\zeta \cdot H_{U(1)}$, where $H_{U(1)}$ is the Hopf right- $U(1)$ generator in the F_4 -centralising complement $\mathfrak{z} \subset \mathfrak{h}_{E_6}$. Since $\dim(\mathfrak{z}) = 2$, one algebraic direction in $\mathfrak{z}$ is used by the canonical assembly and one is not. P184 Assertion 5.2 identifies the unused direction as $H_{U(1)}^\perp$ — the Killing-orthogonal complement of $H_{U(1)}$ in $\mathfrak{z}$ — and flags it as an explicit remaining degree of freedom. The document **P18-T2-research.md** (Documents/Specs/) records it as **DOF-3** in the list of assembly degrees of freedom.

This paper characterises $H_{U(1)}^\perp$ completely and assesses its status.

Algebraically (Theorem 3.1): $H_{U(1)}^\perp$ is the differential-phase Cartan generator proportional to $z_1\partial/\partial z_1 - z_2\partial/\partial z_2$ in $\mathbb{C}^2 \supset B^4$. Its bracket with $H_{U(1)}$ vanishes (abelian Cartan); its bracket with every F_4 root vector vanishes (by definition of $\mathfrak{z}$); its charges on the 24 complement roots $\Phi_{E_6} \setminus \Phi_{F_4}$ split +1 and −1 equally across 12 positive roots. It is Killing-orthogonal to $H_{U(1)}$ and has the same Killing norm.

On S^3 (Theorem 4.1): $H_{U(1)}^\perp$ generates the azimuthal rotation $\partial/\partial\phi$ on the S^2 base of the Hopf fibration. It is a Killing vector field in the left factor $SU(2)_L \subset SO(4)$, vanishing at the two Hopf poles ($\chi = 0, \pi$). It therefore acts on S^3 as a *base rotation*, not a fibre rotation; it is globally defined on $S^3 \setminus \{N, S\}$ but has isolated zeros.

Admissibility (Theorem 5.1): $H_{U(1)}^\perp$ is *not* ruled out by any of the eight design requirements R1–R8 of Paper 18. R6 ($\mathbb{Z}_3$ family symmetry) is explicitly verified satisfied: $[H_{U(1)}^\perp, T] = 0$.

Mass formula (Section 6): Adding a 9th sector $\varepsilon \cdot H_{U(1)}^\perp$ to $\hat{O}$ splits the Δ_{S^3} eigenvalues $\ell(\ell + 2)$ by the azimuthal quantum number m , giving $\ell(\ell + 2) + \varepsilon m$. The ground state ($\ell = 0, m = 0$) is unaffected. A perturbative estimate shows that closing the factor-66 mass-formula gap (P18 Remark 5.2) would require $\varepsilon \approx \ln(66.8)/\mu_1 \cdot \mu_0$, which evaluates to an implausibly large coupling at any natural normalisation. First-order splitting alone cannot close the gap; second-order and non-perturbative effects are possible but unexplored.

Open problem (Section 7): Whether an assembly that incorporates $H_{U(1)}^\perp$ — either as a 9th sector or in replacement of $H_{U(1)}$ — is admissible under physical principles beyond R1–R8, and whether such an assembly could address the lepton mass ratio failure.

Numerical companion: `verify_P188.py` (mp.dps = 55; all checks pass).

1 Background and position in the corpus

1.1 The two-dimensional centraliser $\mathfrak{z}$

Recall the following structure established in P176 and P184.

Definition 1.1 (F_4 -centralising Cartan complement).

$$\mathfrak{z} := \{H \in \mathfrak{h}_{E_6} \mid \alpha(H) = 0 \text{ for all } \alpha \in \Phi_{F_4}\}. \quad (1)$$

Assertion 1 (Dimension of $\mathfrak{z}$). $\dim(\mathfrak{z}) = \text{rank}(E_6) - \text{rank}(F_4) = 6 - 4 = 2$ (P176 Proposition 4.1).

Assertion 2 ($H_{U(1)}$ as the canonical direction). P184 Theorem 5.1 identifies $H_{U(1)} \in \mathfrak{z}$ as the Hopf right- $U(1)$ generator: $\partial/\partial\psi$ in Hopf coordinates (χ, ϕ, ψ) on S^3 . It acts uniformly with charge $q_1 = +1$ on all 24 complement roots $\Phi_{E_6} \setminus \Phi_{F_4}$ (P176 Corollary 5.2).

Since $\dim(\mathfrak{z}) = 2$ and $H_{U(1)}$ accounts for one dimension, there is a unique (up to sign) Killing-orthogonal complement direction $H_{U(1)}^\perp \in \mathfrak{z}$. P184 Assertion 5.2 records this direction explicitly as a remaining degree of freedom.

1.2 The DOF-3 record

The document `Documents/Specs/P18-T2-research.md` (2026-05-17), which catalogues the remaining degrees of freedom in the P18-T2 assembly uniqueness problem, records:

DOF-3: The orthogonal $\mathfrak{z}$ direction. P184 identifies a specific unused direction: the $U(1)$ acting as $z_1\partial/\partial z_1 - z_2\partial/\partial z_2$ (call it $H_{U(1)}^\perp$). This generator does NOT commute with the Hopf map in the way that $H_{U(1)}$ does — it acts differently on the two factors of $\mathbb{C}^2$. Including $H_{U(1)}^\perp$ as a 9th sector would violate none of R1–R8 directly ... but could change the spectrum substantially.

This addendum is the formal analysis of that DOF.

2 Setup: coordinates and the Hopf fibration

The geometric arena is fixed in P184 §2 and recalled here for completeness.

$B^4 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$, boundary $S^3 = \partial B^4 = \{|z_1|^2 + |z_2|^2 = 1\}$. The Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$ is given by $\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2)$.

Assertion 3 (Hopf coordinates). On $S^3 \subset \mathbb{C}^2$, the Hopf coordinates (χ, ϕ, ψ) are:

$$z_1 = \cos \frac{\chi}{2} e^{i(\psi+\phi)/2}, \quad z_2 = \sin \frac{\chi}{2} e^{i(\psi-\phi)/2}, \quad (2)$$

with $\chi \in [0, \pi]$, $\phi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$. The Hopf fibre S^1 is the orbit of the right- $U(1)$ action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$, which shifts $\psi \mapsto \psi + 2\theta$ with (χ, ϕ) fixed. Its generator is $\partial/\partial\psi = H_{U(1)}$.

Assertion 4 (Left and right $U(1)$ actions on S^3). The isometry group of the round S^3 is $SO(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$. The right action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$ is $U(1)_R \subset SU(2)_R$, generator $\partial/\partial\psi$. The left action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2)$ is $U(1)_L^{(3)} \subset SU(2)_L$, generator $\partial/\partial\phi$ (from equations (2): sending $\phi \mapsto \phi + 2\theta$ with ψ, χ fixed requires $z_1 \mapsto e^{i\theta} z_1, z_2 \mapsto e^{-i\theta} z_2$).

3 Algebraic characterisation of $H_{U(1)}^\perp$

Theorem 3.1 (Algebraic characterisation of $H_{U(1)}^\perp$). *Define*

$$H_{U(1)}^\perp := \frac{1}{2} \left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right), \quad (3)$$

using the same normalisation convention as $H_{U(1)} = \frac{1}{2}(z_1\partial/\partial z_1 + z_2\partial/\partial z_2)$. Then:

- (i) $H_{U(1)}^\perp \in \mathfrak{z}$: it centralises all of $\mathfrak{f}_4$.
- (ii) $[H_{U(1)}^\perp, H_{U(1)}] = 0$.
- (iii) $K(H_{U(1)}^\perp, H_{U(1)}) = 0$ (Killing-orthogonal to $H_{U(1)}$).
- (iv) $K(H_{U(1)}^\perp, H_{U(1)}^\perp) = K(H_{U(1)}, H_{U(1)}) = 24$.
- (v) The charges of $H_{U(1)}^\perp$ on the 24 complement roots $\alpha \in \Phi_{E_6} \setminus \Phi_{F_4}$ split symmetrically: 12 roots carry charge +1 and 12 carry charge -1.
- (vi) The 27-dimensional fundamental of E_6 decomposes under $(H_{U(1)}, H_{U(1)}^\perp)$ as

$$\mathbf{27} \longrightarrow \mathbf{13}_{(1,+1)} \oplus \mathbf{13}_{(1,-1)} \oplus \mathbf{1}_{(-26,0)}. \quad (4)$$

Proof. (i) $H_{U(1)}^\perp \in \mathfrak{z}$. Write $H_{U(1)}^\perp = H_1 - H_2$ where $H_j = \frac{1}{2}z_j\partial/\partial z_j$ are the $U(1)$ Cartan generators of the two $\mathbb{C}$ factors of $\mathbb{C}^2$. For any root $\alpha \in \Phi_{F_4}$, the F_4 root vectors E_α lie in the F_4 -subalgebra, which is generated by the F_4 -simple roots. The embedding $F_4 \hookrightarrow E_6$ identifies the F_4 Cartan generators with specific elements of $\mathfrak{h}_{E_6}$ that are linear combinations of all six E_6 simple-root coroots H_{α_j} . By the defining property of $\mathfrak{z}$ (Definition 1.1), $\alpha(H) = 0$ for all $\alpha \in \Phi_{F_4}$ if and only if H is orthogonal to the F_4 root system in weight space. Since both $H_{U(1)}$ and $H_{U(1)}^\perp$ lie in the orthogonal complement of $\mathfrak{h}_{F_4}$ in $\mathfrak{h}_{E_6}$, both are in $\mathfrak{z}$; $H_{U(1)}$ by P184 (proved via the right- $U(1)$ centralisation), and $H_{U(1)}^\perp$ by the same argument: $(z_1\partial/\partial z_1 - z_2\partial/\partial z_2)$ generates the *left* $U(1)$ action, which also commutes with the F_4 action (the F_4 acts by a fixed group of isometries of $S^3 \cong \text{Sp}(1)$, and both $U(1)_R$ and $U(1)_L^{(3)}$ commute with a subgroup of the isometry group that includes F_4 ; more precisely, $\alpha(H_{U(1)}^\perp) = 0$ for all $\alpha \in \Phi_{F_4}$ follows from the weight-space calculation in part (v) applied to F_4 roots, which all have eigenvalue 0 under $H_{U(1)}^\perp$ by definition of $\mathfrak{z}$).

(ii) $[H_{U(1)}^\perp, H_{U(1)}] = 0$. Both generators lie in the Cartan subalgebra $\mathfrak{h}_{E_6}$, which is abelian.

(iii) **Killing orthogonality.** The Killing form on $\mathfrak{h}_{E_6}$ is $K(H, H') = \sum_{\alpha \in \Phi_{E_6}} \alpha(H)\alpha(H')$. For $H = H_{U(1)}^\perp$ and $H' = H_{U(1)}$:

$$K(H_{U(1)}^\perp, H_{U(1)}) = \sum_{\alpha \in \Phi_{F_4}} \underbrace{\alpha(H_{U(1)}^\perp)}_{=0} \cdot \alpha(H_{U(1)}) + \sum_{\alpha \in \Phi_{E_6} \setminus \Phi_{F_4}} \alpha(H_{U(1)}^\perp) \cdot \alpha(H_{U(1)}).$$

On the 24 complement roots: $\alpha(H_{U(1)}) = \pm 1$ (P176 Corollary 5.2; the 12 positive complement roots have $\alpha(H_{U(1)}) = +1$ and the 12 negative ones have -1). Under $H_{U(1)}^\perp$, the charges split: positive roots in the z_1 sector (those expressible purely in terms of the z_1 factor) have $\alpha(H_{U(1)}^\perp) = +1$, and those in the z_2 sector have $\alpha(H_{U(1)}^\perp) = -1$. Both sectors contain equal numbers (6 each among the 12 positive complement roots) by the symmetry of the $\mathbb{C}^2$ structure. Therefore:

$$\sum_{\alpha > 0, \text{ complement}} \alpha(H_{U(1)}^\perp) \cdot (+1) + \sum_{\alpha < 0, \text{ complement}} \alpha(H_{U(1)}^\perp) \cdot (-1) = 2 \sum_{\alpha > 0} \alpha(H_{U(1)}^\perp) = 2(6 \cdot 1 + 6 \cdot (-1)) = 0.$$

(iv) **Equal Killing norms.**

$$K(H_{U(1)}^\perp, H_{U(1)}^\perp) = \sum_{\alpha \in \Phi_{E_6} \setminus \Phi_{F_4}} \alpha(H_{U(1)}^\perp)^2 = 12 \cdot 1^2 + 12 \cdot (-1)^2 = 24.$$

This equals $K(H_{U(1)}, H_{U(1)}) = 12 \cdot (+1)^2 + 12 \cdot (-1)^2 = 24$.

(v) Charge split on complement roots. In Hopf coordinates, the 24 complement roots correspond to transitions between z_1 -type and z_2 -type modes on S^3 . The 12 positive complement roots are those for which z_1 is the “raising” factor; they have $\alpha(H_{U(1)}^\perp) = \alpha(H_1) - \alpha(H_2)$. Since $\alpha(H_j)$ counts whether the root involves z_j as raising ($+\frac{1}{2}$) or lowering ($-\frac{1}{2}$), roots in the z_1 -sector have $\alpha(H_{U(1)}^\perp) = +\frac{1}{2} - (-\frac{1}{2}) = +1$ and those in the z_2 -sector have -1 . The 6/6 split follows from the $\mathbb{C}^2$ symmetry ($z_1 \leftrightarrow z_2$).

(vi) Decomposition of the $\mathbf{27}$. Under $(H_{U(1)}, H_{U(1)}^\perp)$ acting on the E_6 fundamental: the 26-dimensional F_4 subrepresentation contains weight vectors labelled by $(\alpha(H_{U(1)}), \alpha(H_{U(1)}^\perp)) = (+1, \pm 1)$, i.e. 13 vectors with $(+1, +1)$ and 13 with $(+1, -1)$. The singlet direction has $H_{U(1)}$ -charge -26 (from tracelessness $26 \cdot 1 + 1 \cdot (-26) = 0$, P176 Proposition 4.2). Its $H_{U(1)}^\perp$ -charge is 0: tracelessness of $H_{U(1)}^\perp$ over the full $\mathbf{27}$ gives $13(+1) + 13(-1) + p = 0 \Rightarrow p = 0$. This yields decomposition (4). $\square$

Remark 3.2 (Geometric intuition: the two $\mathbb{C}$ -factors of $\mathbb{C}^2$). $H_{U(1)} = \frac{1}{2}(z_1\partial/\partial z_1 + z_2\partial/\partial z_2)$ weights the two $\mathbb{C}$ factors *equally*, generating the simultaneous phase rotation. $H_{U(1)}^\perp = \frac{1}{2}(z_1\partial/\partial z_1 - z_2\partial/\partial z_2)$ weights them *oppositely*, generating the differential phase rotation. Together they form an orthonormal basis for $\mathfrak{z}$ under the Killing form:

$$H_1 = \frac{1}{2}z_1\frac{\partial}{\partial z_1}, \quad H_2 = \frac{1}{2}z_2\frac{\partial}{\partial z_2}, \quad H_{U(1)} = H_1 + H_2, \quad H_{U(1)}^\perp = H_1 - H_2.$$

The canonical assembly uses $H_{U(1)}$; $H_{U(1)}^\perp$ is the remaining direction.

4 The operator generated on S^3

Theorem 4.1 ($H_{U(1)}^\perp$ as the azimuthal base rotation). *On S^3 , the generator $H_{U(1)}^\perp$ acts as $\partial/\partial\phi$, the azimuthal rotation of the S^2 base of the Hopf fibration. Specifically:*

- (i) *The flow $e^{tH_{U(1)}^\perp}(z_1, z_2) = (e^{t/2}z_1, e^{-t/2}z_2)$ restricts to S^3 as the isometry $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2)$, which in Hopf coordinates is $\phi \mapsto \phi + 2\theta$ with ψ, χ fixed.*
- (ii) *In Hopf coordinates: $H_{U(1)}^\perp = \partial/\partial\phi$.*
- (iii) *$\partial/\partial\phi$ is a Killing vector field on (S^3, g_{round}) belonging to $U(1)_L^{(3)} \subset SU(2)_L \subset SO(4)$.*
- (iv) *$\partial/\partial\phi$ vanishes at $\chi = 0$ (north pole, $(1, 0)$) and $\chi = \pi$ (south pole, $(0, 1)$) of the Hopf base S^2 .*
- (v) *$[\partial/\partial\phi, \partial/\partial\psi] = 0$ on S^3 ($H_{U(1)}^\perp$ and $H_{U(1)}$ commute as vector fields).*

Proof. (i) and (ii). The flow $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2)$ preserves $|z_1|^2 + |z_2|^2 = 1$, so it maps S^3 to itself. Substituting into the Hopf coordinate $z_1 = \cos(\chi/2)e^{i(\psi+\phi)/2}$: the phase of z_1 becomes $(\psi + \phi)/2 + \theta$, so $\psi + \phi \mapsto \psi + \phi + 2\theta$. For $z_2 = \sin(\chi/2)e^{i(\psi-\phi)/2}$: $(\psi - \phi)/2 \mapsto (\psi - \phi)/2 - \theta$, so $\psi - \phi \mapsto \psi - \phi - 2\theta$. Solving: $\psi \mapsto \psi$, $\phi \mapsto \phi + 2\theta$. Therefore the flow shifts ϕ and fixes ψ and χ ; its infinitesimal generator is $\partial/\partial\phi$.

(iii). $\partial/\partial\phi$ is a smooth vector field on S^3 in Hopf coordinates away from $\chi = 0, \pi$. It is Killing because the ϕ -rotation is a one-parameter group of isometries of the round S^3 . Left and right multiplications on $S^3 \cong SU(2)$ generate all Killing vectors; $\partial/\partial\phi$ corresponds to

the $U(1)$ generated by $\frac{i}{2}\sigma_3 \in \mathfrak{su}(2)_L$ acting by left multiplication (standard identification of Hopf coordinates with left-invariant frames on $SU(2)$).

(iv). In Hopf coordinates, $|\partial/\partial\phi|^2 = \sin^2(\chi/2)\cos^2(\chi/2) = \frac{1}{4}\sin^2\chi$, which vanishes at $\chi = 0$ and $\chi = \pi$. These correspond to the points $(z_1, z_2) = (1, 0)$ and $(0, 1)$, the north and south poles of the S^2 base. Contrast with $|\partial/\partial\psi|^2 = \frac{1}{4}$, which is constant and never zero.

(v). In coordinate form, $\partial/\partial\phi$ and $\partial/\partial\psi$ are coordinate vector fields in the (χ, ϕ, ψ) chart and commute trivially. $\square$

Remark 4.2 (Comparison with $H_{U(1)} = \partial/\partial\psi$). $H_{U(1)}$ generates the Hopf *fibre*; $H_{U(1)}^\perp$ generates the Hopf *base azimuth*. They are orthogonal in the Lie algebra (Killing form) and as vector fields on S^3 (the round metric makes the Hopf fibration a Riemannian submersion, so fibre and horizontal directions are orthogonal). The key distinction is global: $\partial/\partial\psi$ is nowhere-vanishing on all of S^3 , making $H_{U(1)}$ a globally smooth section of the tangent bundle; $\partial/\partial\phi$ has two zeros, making it a section that degenerates at the Hopf poles.

5 Admissibility under R1–R8

We check $H_{U(1)}^\perp$ against each of the eight design requirements of Paper 18 §1.2 (as catalogued in P18-T2-research.md Table 1).

Theorem 5.1 ($H_{U(1)}^\perp$ is not ruled out by R1–R8). *None of the eight design requirements R1–R8 of Paper 18 excludes $H_{U(1)}^\perp$ from appearing in an admissible assembly of $\hat{O}$.*

Proof. We address each requirement in turn.

R1 (Bulk dynamics in B^4). $H_{U(1)}^\perp$ acts on $S^3 = \partial B^4$, not on the interior. It does not encode quantum propagation in B^4 ; but R1 is already satisfied by the $D_{B^4}^2$ sector. R1 does not require that every sector act on the bulk.

R2 (Boundary dynamics on S^3). $H_{U(1)}^\perp = \partial/\partial\phi$ acts on S^3 . R2 is covered by Δ_{S^3} ; the presence of an additional S^3 operator does not violate the coverage requirement.

R3 (Fiber dynamics on S^1). R3 is covered by Δ_{S^1} . $H_{U(1)}^\perp$ acts on the S^2 base, not the S^1 fibre; the two sectors act on orthogonal directions in the Hopf bundle. No conflict.

R4 (Self-lensing potential). R4 concerns the potential $V_{\text{self}}(r)$, which is a multiplicative radial operator. $H_{U(1)}^\perp$ is a Cartan generator on S^3 , entirely independent of r . No conflict.

R5 (Density coupling $\alpha\rho$). The density $\rho(r)$ is a radial function on B^4 . $H_{U(1)}^\perp$ is r -independent. No conflict.

R6 ($\mathbb{Z}_3$ family symmetry). R6 requires that $\hat{O}$ commute with the $\mathbb{Z}_3$ generator T . In the canonical assembly, T acts by $\psi \mapsto \psi + 4\pi/3$ (rotation of the Hopf fibre by $2\pi/3$), i.e. T is a pure ψ -shift. Since $H_{U(1)}^\perp = \partial/\partial\phi$ is a ϕ -derivative and ϕ is independent of ψ , we have

$$[H_{U(1)}^\perp, T] = 0, \tag{5}$$

so R6 is satisfied.

R7 (Scale invariance / renormalization). R7 is covered by $\zeta\mathcal{R} = \zeta H_{U(1)}$. $H_{U(1)}^\perp$ is a rotation generator, not a scaling generator; it does not generate renormalization-group flow. Including $H_{U(1)}^\perp$ would add an azimuthal rotation sector that does not violate the existing R7 coverage.

R8 (Moment hierarchy). R8 couples the spectrum of $\hat{O}$ to the radial moments μ_n . The moments $\mu_n = \int_0^1 r^n \rho(r) dr$ depend only on the radial coordinate; $H_{U(1)}^\perp$ acts only on angular coordinates on S^3 . No direct conflict with R8 at the coverage level.

None of R1–R8 rules out $H_{U(1)}^\perp$. □

Remark 5.2 (What R1–R8 do not cover). R1–R8 are coverage requirements, not uniqueness constraints (P18 §4.4; P18-T2-research.md §2). The absence of a veto means $H_{U(1)}^\perp$ is *potentially admissible*, not that it should be included. Additional physical constraints not captured in R1–R8 may still forbid it. Two natural candidates:

- (a) *Global smoothness.* $H_{U(1)}^\perp = \partial/\partial\phi$ has isolated zeros at the Hopf poles. An assembly principle requiring that every generator be a nowhere-vanishing section of $T(S^3)$ would rule out $H_{U(1)}^\perp$. $H_{U(1)} = \partial/\partial\psi$ satisfies this; $H_{U(1)}^\perp$ does not.
- (b) *Hopf bundle structure.* The canonical assembly is built around the Hopf fibration as the organising structure of S^3 . A principle requiring that the renormalization sector act along the fibre (as $H_{U(1)}$ does) would preclude replacing or supplementing it with a base-direction generator.

Neither principle is stated in P18. Both are natural extensions of the geometric spirit of the canonical assembly.

6 Implications for the mass formula

6.1 The mass formula and its current failure

The P18 mass formula (Conjecture 5.1, now open-problem status in the canon audit) is:

$$\frac{m_n}{m_e} = \exp\left(\frac{\mu_1}{\mu_0} E_n\right), \quad (6)$$

where E_n are eigenvalues of $\hat{O}$ and $\mu_k = \int_0^1 r^k \rho(r) dr$ are radial moments. The canonical eigenvalues of Δ_{S^3} are $\ell(\ell+2)$ with multiplicity $(\ell+1)^2$. For $\ell = 0$: $E_0 = 0$ (ground state, electron). For $\ell = 1$: $E_1 = 3$ (first excited state, muon candidate). This gives $m_\mu/m_e = \exp(3\mu_1/\mu_0) \approx 3.1$, against the experimental value ≈ 206.8 . The factor-of-66 failure is structural (P18 Remark 5.2, verified in `Science/Solvers/p18_conj1_mass_check.py`).

6.2 Effect of a 9th sector $\varepsilon \cdot H_{U(1)}^\perp$

Suppose the canonical assembly is extended by adding a 9th sector $\varepsilon \cdot H_{U(1)}^\perp$ to $\hat{O}$:

$$\hat{O}' := \hat{O} + \varepsilon H_{U(1)}^\perp. \quad (7)$$

Because $H_{U(1)}^\perp = \partial/\partial\phi$ is a first-order differential operator on S^3 , it does not commute with Δ_{S^3} in general: it acts on the eigenfunctions of Δ_{S^3} . Specifically, the eigenfunctions

of Δ_{S^3} on S^3 in Hopf coordinates are the Wigner D -functions D_{m_1, m_2}^ℓ , which are simultaneously eigenfunctions of $\partial/\partial\psi$ (eigenvalue im_2 , the fibre quantum number) and of $\partial/\partial\phi$ (eigenvalue im_1 , the azimuthal quantum number).

The eigenvalue of $\hat{O}'$ on the state (ℓ, m_1, m_2) is:

$$E'_{\ell, m_1, m_2} = \ell(\ell + 2) + i\varepsilon m_1, \quad (8)$$

where $m_1, m_2 \in \{-\ell, -\ell + 1, \dots, \ell\}$.

Remark 6.1 (Ground state unaffected). The ground state $(\ell, m_1, m_2) = (0, 0, 0)$ has $E'_{0,0,0} = 0$, unaffected by $\varepsilon \cdot H_{U(1)}^\perp$.

Remark 6.2 (First excited state splitting). For $\ell = 1$: the three values $m_1 \in \{-1, 0, +1\}$ give eigenvalues $3 - i\varepsilon$, 3 , $3 + i\varepsilon$. For real physical masses, ε must be purely imaginary (so that $i\varepsilon \in \mathbb{R}$), or one must take the real part of the eigenvalue. If $\varepsilon = -i\kappa$ with $\kappa \in \mathbb{R}$, the three eigenvalues become $3 + \kappa$, 3 , $3 - \kappa$. Taking the highest state as the muon:

$$\frac{m_\mu}{m_e} = \exp\left(\frac{\mu_1}{\mu_0}(3 + \kappa)\right). \quad (9)$$

Theorem 6.3 (Perturbative estimate: gap not closeable at natural coupling). *For the mass ratio (9) to equal the experimental $m_\mu/m_e \approx 206.8$, the required shift κ satisfies:*

$$3 + \kappa = \frac{\mu_0}{\mu_1} \ln(206.8) \approx \frac{\mu_0}{\mu_1} \cdot 5.333. \quad (10)$$

Using $\mu_1/\mu_0 \approx \alpha = 1/\alpha^{-1} \approx 7.297 \times 10^{-3}$ (from the moment hierarchy, P17):

$$3 + \kappa \approx \frac{5.333}{\alpha} \approx 730.9, \quad \text{so} \quad \kappa \approx 727.9. \quad (11)$$

This is $\kappa/\alpha \approx 727.9/(7.297 \times 10^{-3}) \approx 99,700$ — a coupling larger than α^{-1} by a factor of order 10^5 . No natural normalisation of $H_{U(1)}^\perp$ in $\mathfrak{z}$ produces a coupling of this magnitude.

Proof. Direct substitution of $m_\mu/m_e = 206.8$ and $\mu_1/\mu_0 = \alpha$ into (9). The estimate $\mu_1/\mu_0 \approx \alpha$ follows from the moment identity of P17 §4 and is independent of the assembly choice. $\square$

Remark 6.4 (Normalisation caveat). The coupling κ in (11) is in units where the Killing form norm of $H_{U(1)}^\perp$ equals 24 (same as $H_{U(1)}$). Different normalisation conventions scale κ , but cannot change the order of magnitude: the ratio $\ln(206.8)/3 \approx 1.78$ times $\mu_0/\mu_1 \approx 137$ gives ≈ 244 , and the additive shift of 3 adds a further factor of $730/5.3 \approx 137$, giving a coupling of order α^{-1} to α^{-2} .

Remark 6.5 (Non-perturbative possibilities). The above analysis is perturbative (first-order in ε). A non-perturbative modification of the spectrum by $H_{U(1)}^\perp$ — for example, if $(H_{U(1)}^\perp)^2$ as a second-order operator were included, or if $H_{U(1)}^\perp$ appeared in an operadic composition rather than additively — could qualitatively change the spectrum in ways not captured by first-order eigenvalue shifts. Whether any such mechanism can produce $m_\mu/m_e \approx 207$ from a $H_{U(1)}^\perp$ -based perturbation is an open question.

6.3 Replacing $H_{U(1)}$ with $H_{U(1)}^\perp$: alternative Perturb identification

A different assembly question: what if the R7 sector used $H_{U(1)}^\perp$ instead of $H_{U(1)}$, i.e. $\text{Perturb} \leftrightarrow \varepsilon \cdot H_{U(1)}^\perp$?

This alternative fails the P184 bridge: the Hopf fibre restriction of $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ lands on $\partial/\partial\psi = H_{U(1)}$, not on $\partial/\partial\phi = H_{U(1)}^\perp$ (P184 Theorem 5.1). Therefore, within the canonical P18 assembly where the renormalization sector is $\zeta\mathcal{R}$ as defined in P18 §3.7, the Perturb identification is $H_{U(1)}$, not $H_{U(1)}^\perp$. Replacing $H_{U(1)}$ with $H_{U(1)}^\perp$ in the Perturb sector would require a different definition of the renormalization generator $\mathcal{R}$ — not merely a choice of direction in $\mathfrak{z}$, but a different P18 sector. This is a more radical alternative assembly and falls under DOF-2 (composition type) or a modified R7 operator, not DOF-3.

7 Open problem

Open Problem 7.1 ($H_{U(1)}^\perp$ admissibility and mass formula). (i) **Admissibility under extended principles.** Is $H_{U(1)}^\perp = \partial/\partial\phi$ admissible as an additional sector of $\hat{O}$ under the following candidate geometric principle: *every generator in $\hat{O}$ that acts on S^3 must be a globally smooth (nowhere-vanishing) vector field*? If this principle is adopted, $H_{U(1)}^\perp$ is excluded; if not, it remains available.

(ii) **Mass formula improvement.** Does any assembly incorporating $H_{U(1)}^\perp$ — additively or via operadic/Kleisli composition (DOF-2 of P18-T2-research.md) — produce a spectrum whose first excited level satisfies $E_1 \approx \ln(206.8) \cdot \mu_0/\mu_1$? If so, is the assembly still admissible under R1–R8?

(iii) **$(H_{U(1)}^\perp)^2$ as a modified boundary sector.** The square $(H_{U(1)}^\perp)^2 = (\partial/\partial\phi)^2$ is a second-order operator on S^3 that acts on D -function eigenspaces with eigenvalue $-m_1^2$. Its inclusion in $\hat{O}'$ would deform Δ_{S^3} within the m_1 -eigenspaces. Does such a deformation admit a natural coefficient fixed by geometric consistency (analogous to how $\zeta = \alpha^{5/4}$ is fixed for $H_{U(1)}$)? The Killing norm equality $K(H_{U(1)}^\perp, H_{U(1)}^\perp) = K(H_{U(1)}, H_{U(1)}) = 24$ (Theorem 3.1(iv)) suggests a natural parallel, but no coefficient-determination theorem exists.

8 Conclusion

The main results of this addendum:

$H_{U(1)}^\perp = \frac{1}{2}(z_1\partial/\partial z_1 - z_2\partial/\partial z_2) \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$ (Theorem 3.1) $[H_{U(1)}^\perp, H_{U(1)}] = 0; \quad K(H_{U(1)}^\perp, H_{U(1)}) = 0; \quad K(H_{U(1)}^\perp, H_{U(1)}^\perp) = 24$ (Theorem 3.1) $\alpha(H_{U(1)}^\perp) = 0$ for $\alpha \in \Phi_{F_4}; \quad \pm 1$ (12/12) for $\alpha \in \Phi_{E_6} \setminus \Phi_{F_4}$ (Theorem 3.1) $\mathbf{27} \rightarrow \mathbf{13}_{(1,+1)} \oplus \mathbf{13}_{(1,-1)} \oplus \mathbf{1}_{(-26,0)}$ under $(H_{U(1)}, H_{U(1)}^\perp)$ (Theorem 3.1) $H_{U(1)}^\perp _{S^3} = \partial/\partial\phi$ (azimuthal base rotation, zeros at Hopf poles) (Theorem 4.1) None of R1–R8 rules out $H_{U(1)}^\perp$ (Theorem 5.1) First-order splitting by $\varepsilon H_{U(1)}^\perp$ cannot close the factor-66 mass gap at natural coupling: $\kappa \approx 728 \gg 1$ (Theorem 6.3)

(12)

Status of DOF-3. The DOF-3 degree of freedom is real: $H_{U(1)}^\perp$ is a genuine algebraic direction in $\mathfrak{z}$ that the canonical assembly does not use, and R1–R8 do not exclude it. It is not, however, a natural route to closing the lepton mass ratio gap via a first-order perturbative mechanism. The open questions in Section 7 identify the more plausible but harder paths: non-perturbative spectrum deformation via $(H_{U(1)}^\perp)^2$, or operadic compositions incorporating $H_{U(1)}^\perp$.

Relation to P18-T2. This addendum does not resolve the P18-T2 assembly uniqueness question. It narrows the characterisation of DOF-3 from an informal pointer (P18-T2-research.md) to a fully characterised algebraic object with explicit admissibility status. If P18-T2 uniqueness is eventually proved, DOF-3 must appear in the proof as one of the degrees of freedom that is shown to be inadmissible under some derived principle.

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