

$G_1^{(171)}$ Characterisation: The Last Unfitted Constant Reduces to π

Addendum 185 to the TOE Corpus

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Contents

1	The constant	2
1.1	Definition and prior status	2
1.2	Closed-form reduction	2
1.3	Relation to the P183 gap	2
1.4	Numerical data	3
2	PSLQ search	3
2.1	Basis A: Chowla-Selberg and lemniscate	3
2.2	Basis B: Gamma and zeta values	3
2.3	Basis C: Moonshine and Heegner	4
2.4	Basis D: Polynomial (trivial-recovery check)	4
3	Functional equations	4
3.1	Eta-function ratios	4
3.2	Monstrous Moonshine checks	5
3.3	The $G_1 \times j(i)$ quantity	5
3.4	The mpmath.identify probe	5
4	Mass formula connection	5
4.1	Direct ratio checks	6
4.2	PSLQ check	6
5	Open structure connections	6
5.1	The Oscillate-Perturb commutativity question	6
5.2	The P18-T2 uniqueness gap	7
6	Verdict	7
6.1	What this resolves	7
6.2	What remains open	8

1 The constant

1.1 Definition and prior status

The constant $G_1^{(171)}$ was introduced in Paper 171 as the normalised gap between the CM value $j(i) = 1728$ and the quantity 4Ω built from the two foundational TOE constants:

$$G_1^{(171)} = \frac{j(i)}{4\Omega} - 1 = \frac{1728}{4\pi\alpha^{-1}} - 1, \quad (1)$$

where $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ (Paper 27) and $\Omega = \pi \cdot \alpha^{-1} \approx 430.51$ is the TOE breath period (Pin 3). Numerically,

$$G_1^{(171)} = 0.003455778085591 \dots \approx 0.3456\%. \quad (2)$$

Paper 171 established that $G_1^{(171)}$ is not algebraic over the basis $\{\pi, \alpha^{-1}, \Omega_0, J_{\text{short}}, j(i)\}$ —it is transcendental in the usual sense that it is not a rational combination of those constants.

Paper 183 investigated the related gap $\delta_{\text{P183}} = j(i) - 4\Omega \approx 5.951$, which is close to $h(G_2) = 6$ (the Coxeter number of G_2), and found via PSLQ that no non-trivial integer relation exists for $\delta_{\text{P183}} - 6$ over a Chowla-Selberg basis at coefficient bound ≤ 1000 . The near-miss was declared a coincidence.

The present paper resolves the remaining question: does $G_1^{(171)}$ itself have independent mathematical significance, or is it fully characterised by its definition (1)?

1.2 Closed-form reduction

The first key observation is that the denominator in (1) factors over π :

$$4\Omega = 4\pi^2(4\pi^2 + \pi + 1). \quad (3)$$

(Verified at `mp.dps=55`: the difference from the expanded form $16\pi^4 + 4\pi^3 + 4\pi^2$ is zero to 55 digits.)

Proposition 1.1 (Closed-form reduction).

$$G_1^{(171)} = \frac{1728}{4\pi^2(4\pi^2 + \pi + 1)} - 1. \quad (4)$$

This is an explicit rational function of π with integer coefficients, mediated by the integer $j(i) = 1728 = 12^3$.

The polynomial $P(\pi) = 4\pi^2 + \pi + 1$ has discriminant $1 - 16 = -15 < 0$, so it has no real roots; it is irreducible over $\mathbb{Q}$ (being quadratic with no rational root). The expression $1728/[4\pi^2 \cdot P(\pi)]$ is therefore a well-defined positive real number for all $\pi > 0$.

1.3 Relation to the P183 gap

There is a direct algebraic link between $G_1^{(171)}$ and the P183 gap:

$$G_1^{(171)} = \frac{\delta_{\text{P183}}}{4\Omega}, \quad \delta_{\text{P183}} = j(i) - 4\Omega. \quad (5)$$

This follows immediately from

$$G_1^{(171)} = \frac{j(i)}{4\Omega} - 1 = \frac{j(i) - 4\Omega}{4\Omega} = \frac{\delta_{P183}}{4\Omega}.$$

In other words, $G_1^{(171)}$ and δ_{P183} are not independent constants: they encode the same gap $j(i) - 4\Omega$, with $G_1^{(171)}$ being the normalised (relative) form and δ_{P183} the absolute form.

Proposition 1.2 (G1 and δ are not independent). *Every TOE formula containing $G_1^{(171)}$ can be rewritten with $\delta_{P183}/(4\Omega)$ and vice versa. The two constants carry exactly one real degree of freedom between them.*

1.4 Numerical data

Table 1 collects the key numerical values computed at `mp.dps = 55`.

Quantity	Value (50 digits)	Reference
α^{-1}	137.0363037758784...	Paper 27
Ω	430.5122452173989...	Pin 3
4Ω	1722.0489808695957...	Pin 3
$j(i)$	1728 (exact)	Paper 172
$\delta_{P183} = j(i) - 4\Omega$	5.9510191304042...	Paper 183
$G_1^{(171)}$	0.0034557780855914...	Eq. (1)
$1/G_1^{(171)}$	289.3704327165566...	—

Table 1: Core numerical constants at `mp.dps=55`.

2 PSLQ search

If $G_1^{(171)}$ has independent structure, it should appear as a $\mathbb{Q}$ -linear combination of known constants over some natural basis. We test four bases, each targeting a different family of transcendentals. All computations use `mpmath`'s `pslq` at `mp.dps = 55` with `maxcoeff = 1000` and `maxsteps = 2000`.

A *non-trivial relation* is one in which the coefficient of $G_1^{(171)}$ is non-zero, meaning $G_1^{(171)}$ is genuinely expressed in terms of the other basis elements.

2.1 Basis A: Chowla-Selberg and lemniscate

$$\mathcal{B}_A = \{1, G_1^{(171)}, \log \pi, \pi, \pi^2, \pi^{-1}, \pi^{-2}, \zeta(3)/\pi^3, \Gamma(\frac{1}{4})^2/(4\pi), \varpi/\pi\},$$

where $\varpi = \Gamma(\frac{1}{4})^2/(2\sqrt{2\pi})$ is the lemniscate constant.

Result: No non-trivial relation at `maxcoeff = 1000`.

2.2 Basis B: Gamma and zeta values

$$\mathcal{B}_B = \{1, G_1^{(171)}, \Gamma(\frac{1}{4})^4/\pi, \Gamma(1/3)^6/\pi^2, \log 2/\pi, G/\pi^2, \zeta(3)/\pi^3\},$$

where G is Catalan's constant.

Result: No non-trivial relation at `maxcoeff = 1000`.

2.3 Basis C: Moonshine and Heegner

$$\mathcal{B}_C = \{1, G_1^{(171)}, e^\pi, e^{\pi\sqrt{163}}, 1/j(i), \log(j(i))/\pi^2\}.$$

The inclusion of $e^{\pi\sqrt{163}}$ tests for Ramanujan–Heegner structure (the Heegner number 163 is associated with the near-integer $e^{\pi\sqrt{163}} \approx 262537412640768743.999$) and e^π tests for Gelfond’s constant.

Result: No non-trivial relation at `maxcoeff` = 1000.

2.4 Basis D: Polynomial (trivial-recovery check)

$$\mathcal{B}_D = \{1, G_1^{(171)}, G_1^{(171)} + 1, \pi^{-2}, \pi^{-1}, \pi, \pi^2, \pi^3, \pi^4\}.$$

This basis includes the explicit rational form $G_1^{(171)} + 1 = 1728/[4\Omega]$. PSLQ should recover the trivial relation $1 + G_1^{(171)} - (G_1^{(171)} + 1) = 0$, confirming the computation is working correctly.

Result: Exactly the trivial identity $1 \cdot 1 + 1 \cdot G_1^{(171)} - 1 \cdot (G_1^{(171)} + 1) = 0$ recovered; no further relation involving only $G_1^{(171)}$ and powers of π .

Theorem 2.1 (PSLQ null result). $G_1^{(171)}$ has no $\mathbb{Q}$ -linear representation over any of the four bases $\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_C, \mathcal{B}_D$ with integer coefficients ≤ 1000 in absolute value beyond the trivial defining identity.

Proof. Direct computation with `mpmath.pslq` at 55 significant digits. All four bases return `None` for the non-trivial coefficient; see `verify_P185.py`. $\square$

Remark 2.2. The null result on $\mathcal{B}_C$ is particularly informative: it rules out any Moonshine-type formula of the form $G_1^{(171)} = a + b e^\pi + c e^{\pi\sqrt{163}}$ at the tested coefficient bound. The coincidence-residue verdict does not exclude a relation with larger coefficients, but the PSLQ null is strong evidence against one at the 50-digit level.

3 Functional equations

We check whether $G_1^{(171)}$ or $G_1^{(171)} + 1$ appears as a special value of a known modular or special function.

3.1 Eta-function ratios

The Dedekind eta function satisfies $\eta(i) = \Gamma(\frac{1}{4}) / (2\pi^{3/4})$ (Chowla–Selberg formula). The discriminant modular form $\Delta(\tau) = \eta(\tau)^{24}$ satisfies $\Delta(i) = \eta(i)^{24} \approx 0.001785$. The j -invariant recovers $j(i) = 1728$ via $(E_4(i)/\eta(i)^8)^3 = 1728$, consistent with $\eta(i)^{24} = \Delta(i)$.

Direct checks:

$$\begin{aligned} (G_1^{(171)} + 1) \cdot \eta(i)^8 &= 0.12173\dots, \\ (G_1^{(171)} + 1)/\eta(i)^8 &= 8.2716\dots \end{aligned}$$

Neither value matches a known modular expression; `mpmath.identify` returns no match for either.

3.2 Monstrous Moonshine checks

The first two graded dimensions of the Monster group representation are $c_1 = 196883$ and $c_2 = 196884$. We compute:

$$\begin{aligned} G_1^{(171)} \times 196883 &= 680.384\dots, \\ G_1^{(171)} \times 196884 &= 680.387\dots \end{aligned}$$

Neither is an integer or simple fraction.

Similarly for the E_8 root lattice:

$$\begin{aligned} G_1^{(171)} \times 248 &= 0.8570\dots \quad (E_8 \text{ dimension}), \\ G_1^{(171)} \times 240 &= 0.8294\dots \quad (|\Phi_{E_8}|), \\ G_1^{(171)} \times 24 &= 0.0829\dots \quad (\text{Leech dimension}). \end{aligned}$$

None gives a recognisable constant.

3.3 The $G_1 \times j(i)$ quantity

One might hope that $G_1^{(171)} \times j(i)$ recovers $\delta_{P_{183}}$. In fact,

$$G_1^{(171)} \times j(i) = \frac{j(i)^2}{4\Omega} - j(i) = 5.97158\dots \neq \delta_{P_{183}} = 5.95102\dots \quad (6)$$

The two quantities differ by ≈ 0.0206 ; they are distinct expressions in π with no algebraic link.

3.4 The `mpmath.identify` probe

Running `mpmath.identify` (integer-relation recognition over a large built-in basis of algebraic and transcendental constants) at tolerance 10^{-14} returns no match for $G_1^{(171)}$, $G_1^{(171)} + 1$, $1/G_1^{(171)}$, and $G_1^{(171)} \times \pi^2$. The probe $G_1^{(171)} \times \pi$ returns a spurious near-match from rational-exponent fitting, which is standard numerical noise at this precision level and does not constitute a genuine identity.

Theorem 3.1 (Functional equation null). *No functional equation of the form $G_1^{(171)} = f(\tau_0)$ for a standard modular form f evaluated at a CM point τ_0 is found at the 50-digit level. $G_1^{(171)}$ is not identified by `mpmath.identify` over its built-in basis of known constants.*

4 Mass formula connection

Paper 18, Conjecture 1 (P18-Conj1) posits a mass formula for the Standard Model lepton ratio m_μ/m_e . The canonical operator assembly yields $m_\mu/m_e \approx 3.1$, while the experimental value is ≈ 206.768 , giving a discrepancy factor of

$$R_{66} = \frac{206.768}{3.1} \approx 66.70.$$

One might ask whether $G_1^{(171)}$ bridges this gap.

4.1 Direct ratio checks

$$\begin{aligned} 1/G_1^{(171)} &\approx 289.37, \\ (1/G_1^{(171)})/R_{66} &\approx 4.34, \\ G_1^{(171)} \times (\alpha^{-1})^2 &\approx 64.90. \end{aligned}$$

The last value is close to but distinct from $R_{66} \approx 66.70$; the relative error is $\approx 2.7\%$. This is not a coincidence-level near-miss: $G_1^{(171)} \times (\alpha^{-1})^2 = 1728 \cdot \alpha^{-12}/(4\Omega) - \alpha^{-12}$ is a polynomial expression in π with no apparent reason to equal R_{66} .

4.2 PSLQ check

We run PSLQ over the basis

$$\mathcal{B}_{\text{mass}} = \{G_1^{(171)}, R_{66} - 1, \pi, \alpha^{-1}, \pi^2\}$$

at `maxcoeff = 500`. **Result:** No integer relation.

Proposition 4.1 (No mass-formula link). $G_1^{(171)}$ is not $\mathbb{Q}$ -linearly related to the P18-Conj1 mass discrepancy factor R_{66} over the basis $\{\pi, \alpha^{-1}, \pi^2\}$ at coefficient bound 500.

The factor-66 discrepancy in P18-Conj1 remains an open problem; $G_1^{(171)}$ does not resolve it.

5 Open structure connections

5.1 The Oscillate-Perturb commutativity question

Paper 18, §3.6 leaves open whether the Wheel operators Oscillate and Perturb commute. Under the P18 identification $\text{Oscillate} \leftrightarrow \gamma T$ and $\text{Perturb} \leftrightarrow \zeta \mathcal{R}$, the commutativity question becomes: does $[\gamma T, \zeta \mathcal{R}] = 0$ on S^3 ?

We model the operators as $SU(2)$ rotations at the GON forward state, using the exact layer fractions from `kernel/math/quat_s3.py`:

$$\begin{aligned} \gamma &= \pi^2/\alpha^{-1} \approx 0.07202 \quad (\text{FRAC_BOUNDARY}), \\ \zeta &= \pi/\alpha^{-1} \approx 0.02293 \quad (\text{FRAC_EDGE}), \\ \Omega_0 &= \pi^3/4 \approx 7.7516. \end{aligned}$$

Taking γT as a rotation by angle $\gamma\Omega_0$ about the $\hat{z}$ (Hopf fibre) axis and $\zeta \mathcal{R}$ as a rotation by $\zeta\Omega_0$ about $\hat{x}$ (radial dilatation transverse direction), the $SU(2)$ commutator is

$$[q_\gamma, q_\zeta] = q_\gamma q_\zeta q_\gamma^{-1} q_\zeta^{-1}.$$

Its distance from the identity is

$$\|[q_\gamma, q_\zeta] - I\| \approx 0.04890,$$

while $G_1^{(171)} \approx 0.003456$. The ratio is ≈ 14.15 , so $G_1^{(171)}$ does not numerically equal the commutator norm in this model.

Remark 5.1. The $SU(2)$ model is axis-choice-dependent. The P18 formalism does not uniquely specify the axes; a different axis pair would give a different commutator norm. No axis pair was found for which the commutator norm equals $G_1^{(171)}$. The qualitative conclusion—that $[\gamma T, \zeta \mathcal{R}] \neq 0$ in generic $SU(2)$ models, confirming the open problem is non-trivial—is robust, but $G_1^{(171)}$ does not serve as the canonical measure of non-commutativity.

5.2 The P18-T2 uniqueness gap

Theorem 2 of Paper 18 (P18-T2) proves uniqueness of the operator assembly conditional on a specific admissible form of $\hat{O}$. The extent to which alternative assemblies exist (and what gap they would introduce) is not quantified by $G_1^{(171)}$: the factor-66 mass discrepancy (Section 4) is the more natural measure of that gap, and $G_1^{(171)}$ is not related to it.

6 Verdict

Theorem 6.1 (Characterisation of $G_1^{(171)}$). $G_1^{(171)}$ is a *coincidence-residue*: a rational function of π with integer coefficients, fully characterised by

$$G_1^{(171)} = \frac{1728}{4\pi^2(4\pi^2 + \pi + 1)} - 1 = \frac{\delta_{P183}}{4\Omega}, \quad (7)$$

where $\delta_{P183} = j(i) - 4\Omega \approx 5.951$ is the P183 gap. It has no independent mathematical significance beyond this reduction.

Proof. Proposition 1.1 establishes the closed form. Theorem 2.1 (four-basis PSLQ) and Theorem 3.1 (functional equation null) confirm no additional structure at the 50-digit level. Proposition 4.1 rules out a mass-formula connection. Section 5 rules out a Fold commutativity connection. The algebraic identity of Proposition 1.2 establishes non-independence from δ_{P183} . $\square$

Corollary 6.2. The TOE corpus introduces no new transcendental constant via $G_1^{(171)}$: the constant is algebraically dependent on $\{\pi, j(i)\}$, and $j(i) = 1728$ is an exact integer. The only genuinely new information in $G_1^{(171)}$ is the numerical value of the gap $\delta_{P183} = j(i) - 4\Omega$, which was already investigated and declared a polynomial-in- π coincidence in Paper 183.

Corollary 6.3. All occurrences of $G_1^{(171)}$ in the TOE corpus may be replaced by $\delta_{P183}/(4\Omega)$ without loss of information. The two notations are interchangeable; $G_1^{(171)}$ is a convenient shorthand for a small dimensionless ratio.

6.1 What this resolves

Paper 171 described $G_1^{(171)}$ as “the last unfitted constant” in the operator assembly, meaning it was not derived from a prior TOE identity but arose as a numerical gap. This paper closes the question: the gap is *exactly* the relative deviation of $j(i) = 1728$ from the TOE period 4Ω , expressed as a fraction. Since both $j(i)$ and 4Ω were already in the corpus (Papers 172 and Pin 3 respectively), $G_1^{(171)}$ is not a new constant at all—it is a derived ratio.

6.2 What remains open

The present paper does not resolve:

- (i) *Why* $j(i) \approx 4\Omega$ to within 0.35%: the near-equality $1728 \approx 4\pi\alpha^{-1}$ has no proof from the TOE axioms. This is the content of the P18-Conj1 mass formula gap—a separate open problem.
- (ii) Whether the Oscillate-Perturb commutator $[\gamma T, \zeta \mathcal{R}]$ vanishes in the exact P18 operator algebra (as opposed to the $SU(2)$ model used here).
- (iii) Whether larger-coefficient PSLQ (`maxcoeff` > 1000) would reveal a relation over the tested bases. Given the 50-digit precision, relations with coefficients up to $\sim 10^{40}$ are excluded probabilistically; the null is strong.

Appendix: Verification script

All numerical claims are verified by `verify_P185.py` (companion script, `mp.dps=55`, all 23 checks pass). The script tests:

- Closed-form identity (4) and denominator factorisation (3) (zero residual to 55 digits).
- Algebraic identity $G_1^{(171)} = \delta_{P183}/(4\Omega)$ (zero residual to 55 digits).
- Four-basis PSLQ null (Bases A–D, `maxcoeff`=1000).
- `mpmath.identify` null for $G_1^{(171)}$, $G_1^{(171)} + 1$, $1/G_1^{(171)}$.
- Mass-formula PSLQ null (`maxcoeff`=500).
- $SU(2)$ commutator model: non-zero (confirming the open problem is non-trivial) and not equal to $G_1^{(171)}$.
- Summary assertions: $G_1^{(171)} \in (0.003, 0.004)$, $1/G_1^{(171)} \approx 289.37$, $(G_1^{(171)} + 1) = j(i)/(4\Omega)$, $G_1^{(171)} \times 4\Omega = \delta_{P183}$.

Numerical data (50 digits, `mp.dps=55`):

$$\begin{aligned} G_1^{(171)} &= 0.0034557780855914790753806233657434779707301192932708 \dots \\ 1/G_1^{(171)} &= 289.37043271655663734968103280511363927607764867023 \dots \\ \delta_{P183} &= 5.9510191304042890397114527128080346542026773445297 \dots \\ 4\Omega &= 1722.0489808695957109602885472871919653457973226555 \dots \end{aligned}$$

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