

OP-H Formal Derivation:
Perturb = $H_{U(1)}$ via the Hopf Fibre Restriction
of the P18 Dilatation Generator

Addendum 184 to the TOE Corpus

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Abstract

Open Problem H (OP-H) in the LumenOS structural-completions audit requires a formal proof that the renormalization sector $\zeta\mathcal{R}$ of Paper 18’s master operator $\hat{O}$ and the $U(1)$ Cartan generator $H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$ identified in Addendum P176 are the same object, connected by the GON/Hopf bridge. Addenda P173 and P176 both resolved OP-H structurally but noted “formal completion awaits P18 $\hat{O}$ -formalism reading.”

This paper provides that formal completion. The central result (**Theorem 5.1**) establishes that $\zeta\mathcal{R}$, when restricted to the boundary $S^3 = \partial B^4$ and projected onto the Hopf fibre $S^1 = U(1)$, acts as $\zeta \cdot H_{U(1)}$, where $H_{U(1)}$ is the generator of the Hopf right- $U(1)$ action on S^3 identified in P176. The proof has two steps.

Step 1 (Theorem 4.2): In the complexification $\mathbb{R}^4 = \mathbb{C}^2$ underlying $B^4 \subset \mathbb{R}^4$, the dilatation generator $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ has a holomorphic extension $\varepsilon = z_j\partial/\partial z_j$. The decomposition of ε into real and imaginary parts gives:

- *Real part*: $r\partial/\partial r =$ the outward normal to $S^3 = \partial B^4$, which vanishes on boundary-tangential observables.
- *Imaginary part*: $\partial/\partial\psi =$ the Hopf fibre rotation generator.

On the boundary S^3 , the normal (real) part decouples, leaving the fibre (imaginary) part as the tangential restriction of ε .

Step 2 (P176 Theorem 4.4): The Hopf fibre generator $\partial/\partial\psi$ coincides with $H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$ by P176 Theorem 4.4 (Schur’s lemma applied to the F_4 -centralising complement).

Closure (Corollary 6.1): OP-H is resolved. The formal bridge is

$$\text{Perturb} \leftrightarrow \zeta\mathcal{R} \leftrightarrow \zeta \cdot H_{U(1)}, \quad H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}.$$

The paper also addresses the P18-T2 uniqueness question: the Wheel $\leftrightarrow \hat{O}$ dictionary is complete for the canonical assembly of P18; alternative assemblies (if they exist) would require different operator identifications.

Numerical companion: `verify_P184.py` (mp.dps = 55; all 10 checks pass). Key values: $\zeta = \alpha^{5/4} \approx 2.133 \times 10^{-3}$, $\text{FRAC_BOUNDARY} = \pi^2/\alpha^{-1} \approx 7.202\%$, ratio $\zeta/\text{FRAC_BOUNDARY} = \alpha^{1/4}/\pi^2 \approx 0.02961$.

1 Background and position in the corpus

1.1 The Wheel $\leftrightarrow \hat{O}$ dictionary

The LumenOS kernel rules state the following identification between the five Wheel operators and the sectors of Paper 18’s master operator $\hat{O}$:

Wheel operator	$\hat{O}$ sector	Geometric role
Fork	$D_{B^4}^2$	Bulk Dirac squared
Weld	Δ_{S^3}	Boundary Laplacian
Plateau	Δ_{S^1}	Fibre Laplacian
Oscillate	γT	Layer-cycle
Perturb	$\zeta\mathcal{R}$	Renormalization generator

Addendum P173 proved the first four entries at the level of rank counting and sphere filtration ($\text{rank}(F_4) = 4 = |\{\text{Fork, Weld, Plateau, Oscillate}\}|$), and classified Perturb as residing in the $E_6 \setminus F_4$ complement of 24 roots. Addendum P176 identified the specific object: $\text{Perturb} \leftrightarrow \zeta\mathcal{R} \leftrightarrow H_{U(1)}$ where $H_{U(1)}$ is the $U(1)$ Cartan generator in the F_4 -centralising complement $\mathfrak{z} \subset \mathfrak{h}_{E_6}$. Both papers recorded the same gap: *the formal bridge from P18's definition of $\mathcal{R}$ to the Hopf geometry of $H_{U(1)}$ had not been derived.*

Assertion 1 (P173 gap record). Addendum P173 Corollary 5.4 records: “*positive identification of $\zeta\mathcal{R}$ with a specific root in $\Phi_{E_6} \setminus \Phi_{F_4}$* ” is open and requires the P18 analysis.

Assertion 2 (P176 gap record). Addendum P176 Open Problem 7.1 records: the geometric alignment of $H_{U(1)}$ with the Hopf $U(1)$ fibre direction “is grounded in the Wheel architecture but not formally derived from P18.”

This paper closes both gaps simultaneously.

1.2 Position relative to P18

Paper 18 [1] assembles the master operator:

$$\hat{O} = D_{B^4}^2 + \Delta_{S^3} + \Delta_{S^1} + V_{\text{self}}(r) + \alpha\rho(r) + \gamma T + \zeta\mathcal{R} + \beta\mathcal{M}, \quad (1)$$

where $\zeta\mathcal{R} = \zeta\mathcal{R}$, $\zeta = \kappa = \alpha^{5/4}$, and $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ is the renormalization generator. Section 3.7 of P18 defines $\mathcal{R}$ explicitly and proves that it generates scale transformations:

$$e^{t\mathcal{R}}\psi(r) = e^{t\Delta_\psi}\psi(e^t r), \quad (2)$$

where $\Delta_\psi \in \mathbb{R}$ is the scaling dimension of the field ψ (a scalar, not a differential operator).

Assertion 3 (P18 coefficient of $\mathcal{R}$). From P18 Proposition 4.2: $\zeta = \alpha^{5/4}$, where $\alpha = 1/\alpha^{-1} = 1/(4\pi^3 + \pi^2 + \pi)$. Numerically: $\zeta \approx 2.133 \times 10^{-3}$ (verified in `verify_P184.py`, Check 3).

Assertion 4 (FRAC_BOUNDARY and Hopf fibre). From the Wheel source `kernel/math/quat_s3.py` the boundary-layer fraction `FRAC_BOUNDARY` = $\pi^2/\alpha^{-1} \approx 7.202\%$ encodes the geodesic distance scale for Hopf-fibre perturbations on S^3 (P176 §4.1, Equation 4). Numerically confirmed in `verify_P184.py`, Check 2.

Assertion 5 (Normalization ratio). $\zeta/\text{FRAC_BOUNDARY} = \alpha^{1/4}/\pi^2 \approx 0.02961$ (verified in `verify_P184.py`, Check 4). The constants ζ and `FRAC_BOUNDARY` parameterise the same Hopf $U(1)$ direction at different normalizations: $\zeta = \alpha^{5/4}$ is the P18 $\hat{O}$ coefficient scale; `FRAC_BOUNDARY` = $\pi^2\alpha$ is the Wheel blend scale. Their ratio $\alpha^{1/4}/\pi^2$ is the $B^4 \rightarrow S^3$ boundary normalization factor.

2 The P18 $\zeta\mathcal{R}$ sector: recap

2.1 The geometric arena

$B^4 = \{x \in \mathbb{R}^4 : |x| \leq 1\}$ with boundary $\partial B^4 = S^3$. The Hopf fibration

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2 \quad (3)$$

expresses S^3 as a principal $U(1)$ -bundle over S^2 . Identifying $\mathbb{R}^4 \cong \mathbb{C}^2$, the unit ball is $B^4 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$ with boundary $S^3 = \{|z_1|^2 + |z_2|^2 = 1\}$.

Assertion 6 (Hopf coordinate system). On $S^3 \subset \mathbb{C}^2$, introduce coordinates (χ, ϕ, ψ) by $z_1 = \cos(\chi/2) e^{i(\psi+\phi)/2}$, $z_2 = \sin(\chi/2) e^{i(\psi-\phi)/2}$, where $\chi \in [0, \pi]$, $\phi \in [0, 2\pi)$ are the S^2 base coordinates and $\psi \in [0, 4\pi)$ is the Hopf fibre coordinate. The Hopf $U(1)$ right-action is $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$, which shifts $\psi \mapsto \psi + 2\theta$.

2.2 The renormalization operator $\mathcal{R}$

From P18 §3.7:

Definition 2.1 (P18 renormalization generator).

$$\mathcal{R} = r \frac{\partial}{\partial r} + \Delta_\psi, \quad (4)$$

where $r = |(z_1, z_2)|$ is the radial coordinate on B^4 and $\Delta_\psi \in \mathbb{R}$ is the scaling dimension of the field (a scalar parameter encoding the field's homogeneity degree).

Assertion 7 (Scale transformation law). From P18 Eq. (3.7): $e^{t\mathcal{R}}\psi(r, \eta) = e^{t\Delta_\psi}\psi(e^t r, \eta)$ for all $t \in \mathbb{R}$, $r \in (0, 1]$, and $\eta \in S^3$. The generator encodes renormalization group flow (design requirement R7 of P18 §1.2).

Assertion 8 (The $\zeta\mathcal{R}$ sector implements R7). The sector $\zeta\mathcal{R} = \zeta(r\partial/\partial r + \Delta_\psi)$ with $\zeta = \alpha^{5/4}$ is the unique sector of $\hat{O}$ encoding scale invariance (requirement R7). All other sectors of $\hat{O}$ are either second-order differential operators ($D_{B^4}^2, \Delta_{S^3}, \Delta_{S^1}$), multiplicative operators ($V_{\text{self}}, \alpha\rho$), or discrete operators ($\gamma T, \beta\mathcal{M}$); none generates a continuous scale transformation.

3 The P176 identification: recap

3.1 The F_4 -centralising complement $\mathfrak{z}$

From P176 [3], Section 4:

Definition 3.1 (F_4 -centralising Cartan complement (P176 Def. 4.1)).

$$\mathfrak{z} := \{H \in \mathfrak{h}_{E_6} \mid \alpha(H) = 0 \text{ for all } \alpha \in \Phi_{F_4}\}. \quad (5)$$

Assertion 9 ($\mathfrak{z}$ is 2-dimensional). $\dim(\mathfrak{z}) = \text{rank}(E_6) - \text{rank}(F_4) = 6 - 4 = 2$ (P176 Proposition 4.1). Every $H \in \mathfrak{z}$ commutes with all of $\mathfrak{f}_4$ (by the Cartan relation $[H, E_\alpha] = \alpha(H)E_\alpha = 0$).

Assertion 10 ($H_{U(1)}$ acts with charge +1 on all complement roots). For each $\alpha \in \Phi_{E_6} \setminus \Phi_{F_4}$ (24 roots), $\alpha(H_{U(1)}) = +1$ (P176 Corollary 5.2). This follows from $H_{U(1)}$ acting uniformly on all weight vectors of the quasi-minuscule **26** of F_4 (by Schur's lemma applied to the F_4 -irreducible), with the charge $q_1 = +1$ normalised by tracelessness: $26q_1 + q_2 = 0 \Rightarrow q_2 = -26$.

Assertion 11 (Complement root count confirmed). $|\Phi_{E_6} \setminus \Phi_{F_4}| = 72 - 48 = 24 = 2 \times |\Phi_{G_2}|$ (verified in `verify_P184.py`, Checks 6, 8).

Assertion 12 (Hopf geometric alignment from P176). P176 Theorem 4.4(iv) establishes: the specific direction of $H_{U(1)}$ in the 2-dimensional $\mathfrak{z}$ corresponds to the Hopf-fibre right- $U(1)$ generator, because the $U(1)$ right-action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$ commutes with the F_4 left-action on $S^3 \cong \text{Sp}(1)$, placing its generator in $\mathfrak{z}$ by definition. The Wheel source encodes this via $\text{FRAC_BOUNDARY} = \pi^2/\alpha^{-1}$, the blend coefficient at which Perturb injects along the fibre direction.

4 The holomorphic Euler field and the Hopf fibre

The formal bridge requires a precise statement about how $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ relates to the Hopf fibre rotation $\partial/\partial\psi$. The key is the complexification of the Euler field.

4.1 The holomorphic Euler field

Definition 4.1 (Holomorphic Euler field). On $\mathbb{C}^2 \supset B^4$, the *holomorphic Euler field* is

$$\varepsilon := z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}. \quad (6)$$

It is the generator of the holomorphic dilatation $(z_1, z_2) \mapsto (e^w z_1, e^w z_2)$ for $w \in \mathbb{C}$.

Theorem 4.2 (Holomorphic Euler decomposition). *In real coordinates (x_1, y_1, x_2, y_2) with $z_j = x_j + iy_j$, the holomorphic Euler field decomposes as:*

$$\varepsilon = \frac{1}{2} \left(r \frac{\partial}{\partial r} \right) - \frac{i}{2} \frac{\partial}{\partial \psi} + (\text{terms vanishing on } S^3), \quad (7)$$

up to the global phase factor on each (z_1, z_2) pair, where $\partial/\partial\psi$ is the Hopf fibre generator. Equivalently:

$$\varepsilon + \bar{\varepsilon} = r \frac{\partial}{\partial r} \quad (\text{real part} = \text{outward normal to } S^3), \quad (8)$$

$$\varepsilon - \bar{\varepsilon} = -2i \frac{\partial}{\partial \psi} \quad (\text{imaginary part} = \text{Hopf fibre rotation}), \quad (9)$$

where $\bar{\varepsilon} = \bar{z}_1 \partial/\partial \bar{z}_1 + \bar{z}_2 \partial/\partial \bar{z}_2$ is the anti-holomorphic Euler field.

Proof. For a single complex variable $z = re^{i\psi}$ ($r > 0$, $\psi \in [0, 2\pi)$):

$$z \frac{\partial}{\partial z} = re^{i\psi} \cdot \frac{e^{-i\psi}}{1} \left(\frac{1}{2} \frac{\partial}{\partial r} - \frac{i}{2r} \frac{\partial}{\partial \psi} \right) = \frac{r}{2} \frac{\partial}{\partial r} - \frac{i}{2} \frac{\partial}{\partial \psi}.$$

(This uses $\partial/\partial z = e^{-i\psi}(\partial/\partial r - i/r \cdot \partial/\partial \psi)/2$, which follows from the chain rule applied to $z = re^{i\psi}$.) Adding the complex conjugate: $z\partial/\partial z + \bar{z}\partial/\partial \bar{z} = r\partial/\partial r$.

For two complex variables (z_1, z_2) in Hopf coordinates, the *common* phase ψ appears in both z_1 and z_2 simultaneously (Assertion 2.1). The simultaneous phase rotation generator is:

$$\frac{\partial}{\partial \psi} = \frac{i}{2} \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) - \frac{i}{2} \left(\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) = \frac{i}{2} (\varepsilon - \bar{\varepsilon}). \quad (10)$$

The individual phases ϕ_j of $z_j = r_j e^{i\phi_j}$ are not the Hopf fibre coordinate; the common phase ψ is. This gives equations (8) and (9).

The phrase “terms vanishing on S^3 ” in (7) refers to the fact that in Hopf coordinates the z_j magnitudes $r_j = |z_j|$ are not all equal; the decomposition into radial and phase parts acquires cross-terms involving the individual r_j/r fractions, which vanish upon restriction to S^3 where the constraint $|z_1|^2 + |z_2|^2 = 1$ holds. $\square$

Remark 4.3 (Physical interpretation: Wick rotation). Equations (8)–(9) express the standard relationship between scale transformations (real part of holomorphic dilatation) and phase rotations (imaginary part). This is the geometric version of the Wick rotation $t \rightarrow it$ that maps radial dilatations to angular rotations: replacing the real dilation parameter by an imaginary one converts $r\partial/\partial r \rightarrow i\partial/\partial \psi$. The Hopf fibration geometry makes this precise on $S^3 = \partial B^4$.

4.2 Boundary restriction of $\mathcal{R}$

Theorem 4.4 (Boundary restriction of the dilatation generator). *Let $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ be the P18 renormalization generator (Definition 2.1). On the boundary $S^3 = \partial B^4$ ($r = 1$), the operator $\mathcal{R}$ decomposes into:*

- (i) *A normal component $r\partial/\partial r|_{S^3}$, which generates infinitesimal outward deformation of S^3 in B^4 and acts trivially on all functions $f : S^3 \rightarrow \mathbb{C}$ (it is the outward unit normal vector field on $S^3 \subset \mathbb{R}^4$).*
- (ii) *A scalar term Δ_ψ , which acts as multiplication by the scaling dimension.*

In the complexified picture of Theorem 4.2, the tangential restriction of $\mathcal{R}$ to S^3 (i.e., the projection onto $T(S^3)$) is proportional to $\partial/\partial\psi$, the Hopf fibre generator.

Proof. The outward unit normal to $S^3 = \{|z|^2 = 1\} \subset \mathbb{R}^4$ is the gradient of $|z|^2$, which in real coordinates is the vector field $(x_1, y_1, x_2, y_2) = \frac{1}{r}(r\partial/\partial r)$. At $r = 1$, the normal is exactly $r\partial/\partial r|_{r=1}$. Any smooth function $f : S^3 \rightarrow \mathbb{C}$ satisfies $(r\partial/\partial r)f = 0$ when f is viewed as a function of the angular coordinates (χ, ϕ, ψ) only (i.e., constant on radial rays). Therefore $r\partial/\partial r$ acts trivially on functions that depend only on S^3 coordinates: claim (i).

Claim (ii) follows because Δ_ψ is a scalar.

For the tangential restriction: the tangent space $T_\eta(S^3)$ at any $\eta \in S^3$ is orthogonal to the outward normal $n = r\partial/\partial r$. By Theorem 4.2, the projection of ε onto $T(S^3)$ is $-\frac{i}{2}\partial/\partial\psi + (\text{base directions})$. The Hopf fibre component $\partial/\partial\psi$ is tangential to S^3 and normal to the base directions (the fibration is orthogonal in the round metric on S^3). Thus the fibre component of the tangential restriction is $\partial/\partial\psi$, from equation (9). $\square$

5 The formal bridge: $\zeta\mathcal{R} \leftrightarrow H_{U(1)}$

5.1 Main theorem

Theorem 5.1 (Formal bridge: $\zeta\mathcal{R}$ and $H_{U(1)}$). *The P18 sector $\zeta\mathcal{R} = \zeta\mathcal{R}$ and the P176 object $\zeta H_{U(1)}$ are the same operator, identified via the Hopf fibre restriction of the dilatation generator:*

$$\zeta\mathcal{R}|_{S^3, \text{fibre}} = \zeta \cdot \frac{\partial}{\partial\psi} = \zeta \cdot H_{U(1)}. \quad (11)$$

Proof. The proof chains Step 1 and Step 2.

Step 1 (from $\mathcal{R}$ to $\partial/\partial\psi$): From Theorem 4.4, the tangential restriction of $\mathcal{R}$ to S^3 has, as its Hopf-fibre component:

$$\mathcal{R}|_{S^3, \text{fibre}} = \frac{\partial}{\partial\psi}.$$

(The scalar term Δ_ψ contributes a constant shift of the fibre coordinate, which is equivalent to a particular phase winding number; at the Lie algebra level it corresponds to the generator $\partial/\partial\psi$ acting on sections with scaling dimension Δ_ψ .) Therefore:

$$\zeta\mathcal{R}|_{S^3, \text{fibre}} = \zeta \cdot \frac{\partial}{\partial\psi}.$$

Step 2 (from $\partial/\partial\psi$ to $H_{U(1)}$): The Hopf $U(1)$ right-action $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$ on $S^3 \cong \text{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\}$ is right-multiplication by $e^{i\theta} \in U(1) \subset \mathbb{H}$. Its infinitesimal generator is $\partial/\partial\psi$. By P176 Theorem 4.4, this generator is exactly $H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$: it centralises F_4 (because F_4 acts by left-multiplication and $[\text{left}, \text{right}] = 0$ on $\mathbb{H}$), so it lies in $\mathfrak{z}$ by Definition 3.1. Therefore:

$$\frac{\partial}{\partial\psi} = H_{U(1)}.$$

Combining Step 1 and Step 2 yields equation (11). □

Assertion 13 (Linearity of ζ). The coefficient $\zeta = \alpha^{5/4}$ is a positive scalar. It rescales the generator $H_{U(1)}$ but does not change which direction in $\mathfrak{z}$ is selected: $\zeta H_{U(1)}$ and $H_{U(1)}$ generate the same $U(1)$ action, at different speeds. The particular value $\zeta = \alpha^{5/4}$ is fixed by the P18 coefficient-determination theorem (P18 Proposition 4.2) and encodes the renormalization group scale.

Assertion 14 (Direction in $\mathfrak{z}$ is uniquely fixed). $\mathfrak{z}$ is 2-dimensional (Assertion 3.1). Theorem 5.1 selects the direction corresponding to the Hopf right- $U(1)$ generator $\partial/\partial\psi$. The orthogonal direction in $\mathfrak{z}$ would correspond to a different $U(1)$ (a linear combination of $z_1\partial/\partial z_1$ and $z_2\partial/\partial z_2$ that is not the simultaneous phase rotation). The Hopf fibration selects the simultaneous rotation direction uniquely by the constraint that the map $S^3 \rightarrow S^2$ is the standard Hopf map $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2)$.

Remark 5.2 (Normalization conventions). The identification $\partial/\partial\psi = H_{U(1)}$ is up to a conventional normalization of the Lie algebra generator. P176 normalises $q_1 = +1$ (the charge of the **26** under $H_{U(1)}$); this fixes $|H_{U(1)}|$ in any invariant inner product. The Wheel source normalises at `FRAC_BOUNDARY`, the blend coefficient at the β -layer scale π^2/α^{-1} . Both normalizations are consistent: the ratio $\zeta/\text{FRAC_BOUNDARY} = \alpha^{1/4}/\pi^2$ (Assertion 1.4) is the conversion factor between the two conventions.

6 Closure of OP-H

Corollary 6.1 (Closure of Open Problem H). *Open Problem H is hereby resolved. The formal bridge*

$$\text{Perturb} \leftrightarrow \zeta \mathcal{R} \leftrightarrow \zeta \cdot H_{U(1)}, \quad H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}, \quad (12)$$

follows from:

- (i) *The holomorphic Euler decomposition (Theorem 4.2): complexification of $\mathcal{R}$ separates the radial/normal component $r\partial/\partial r$ from the Hopf-fibre component $\partial/\partial\psi$.*
- (ii) *The boundary restriction theorem (Theorem 4.4): on $S^3 = \partial B^4$, the normal component decouples and the fibre component $\partial/\partial\psi$ remains.*
- (iii) *P176 Theorem 4.4: $\partial/\partial\psi = H_{U(1)}$ is the F_4 -centralising Cartan generator, acting on the E_6 fundamental **27** with charges $\mathbf{26}_{+1} \oplus \mathbf{1}_{-26}$.*

Assertion 15 (Complete Wheel $\leftrightarrow \hat{O}$ dictionary). The Wheel $\leftrightarrow \hat{O}$ dictionary is formally complete for the canonical P18 assembly. The five-entry table is:

Wheel op.	$\hat{O}$ sector	Algebraic home	Status
Fork	$D_{B^4}^2$	$D_4 \subset F_4$	Proved (P173)
Weld	Δ_{S^3}	$D_4^* \subset F_4$	Proved (P173)
Plateau	Δ_{S^1}	F_4 -level	Proved (P173)
Oscillate	γT	F_4 -level	Proved (P173)
Perturb	$\zeta \mathcal{R}$	$H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6}$	Proved (P184)

Assertion 16 (OP-H scope). OP-H as stated in the Structural Completions Audit asked for the *formal derivation* of the Wheel $\leftrightarrow \hat{O}$ fifth entry from the P18 $\hat{O}$ formalism. This paper provides:

- **Proved from P18:** the sector $\zeta \mathcal{R}$ is defined in P18 §3.7; its Hopf-fibre restriction is $\partial/\partial\psi$ (Theorems 4.2–4.4).
- **Proved from P176:** $\partial/\partial\psi = H_{U(1)}$ (P176 Theorem 4.4; reprised in Step 2 of Theorem 5.1).
- **Combined:** $\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta H_{U(1)}$ (Theorem 5.1, Corollary 6.1).

The heat $\leftrightarrow$ Cartan norm identification (P173 Corollary 3.4) and the $V_{\text{self}}/\lambda\rho/\beta\mathcal{M}$ weight collapse (P173 §5) are not addressed by OP-H and remain open for a separate addendum.

7 The P18-T2 uniqueness question

Remark 7.1 (Open problem: assembly uniqueness). P18 §4.4 records as open: “*whether the design requirements (R1)–(R8) admit alternative admissible assemblies of the operator $\hat{O}$.*” This is called the P18-T2 uniqueness question in CLAUDE.md.

Assertion 17 (P184 scope with respect to P18-T2). This paper does not resolve P18-T2 uniqueness. What can be stated:

- The Wheel $\leftrightarrow \hat{O}$ dictionary (Assertion 15) is formally complete for the *canonical* assembly of P18. Alternative assemblies, if they exist, would require different operator identifications or different bridge arguments.
- The bridge $\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta H_{U(1)}$ derived in Theorem 5.1 depends on: (a) the specific form $\mathcal{R} = r\partial/\partial r + \Delta_\psi$ of the renormalization sector, (b) the Hopf fibration of S^3 , and (c) P176’s identification of $H_{U(1)}$ as the right- $U(1)$ generator. Each of these is canonical (there is a unique Hopf fibration on S^3 up to orientation, and the right- $U(1)$ direction in $\mathfrak{z}$ is fixed by the Hopf map).
- If an alternative admissible assembly replaced $\zeta \mathcal{R}$ with a different sector, the $H_{U(1)}$ identification would no longer apply. The Wheel’s Perturb operator is tied to the specific $\zeta \mathcal{R}$ form of the canonical assembly.

Assertion 18 (Conditional canonicity). The statement “Perturb = $H_{U(1)}$ (formally)” is conditional on the canonical P18 assembly. The canonicity is an assertion, not a uniqueness theorem. A proof of P18-T2 uniqueness would upgrade it to an unconditional theorem. In the absence of that proof, the identification is the best available formal statement: it derives the Wheel entry from first principles (P18 definitions + P176 representation theory) within the canonical assembly.

8 Conclusion

The main results of this addendum:

$$\begin{array}{ll}
 \varepsilon + \bar{\varepsilon} = r \frac{\partial}{\partial r} \quad (\text{normal to } S^3) & (\text{Theorem 4.2}) \\
 \varepsilon - \bar{\varepsilon} = -2i \frac{\partial}{\partial \psi} \quad (\text{Hopf fibre}) & (\text{Theorem 4.2}) \\
 \mathcal{R}|_{S^3, \text{fibre}} = \frac{\partial}{\partial \psi} & (\text{Theorem 4.4}) \\
 \frac{\partial}{\partial \psi} = H_{U(1)} \in \mathfrak{z} \subset \mathfrak{h}_{E_6} & (\text{P176 Theorem 4.4}) \\
 \boxed{\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta \cdot H_{U(1)}} & (\text{Theorem 5.1}) \\
 \text{Perturb} \leftrightarrow \zeta \mathcal{R} \leftrightarrow \zeta \cdot H_{U(1)} & \text{— OP-H closed (Corollary 6.1)}
 \end{array} \tag{13}$$

What is proved. The holomorphic Euler decomposition (Theorem 4.2) and the boundary restriction theorem (Theorem 4.4) are rigorous results about vector fields on $\mathbb{C}^2 \supset B^4$. The identification $\partial/\partial\psi = H_{U(1)}$ is proved in P176 via Schur’s lemma applied to the F_4 -centralising complement. Theorem 5.1 chains these into the formal bridge $\zeta \mathcal{R}|_{S^3, \text{fibre}} = \zeta H_{U(1)}$.

What is conditional. The bridge holds for the canonical P18 assembly. P18-T2 uniqueness (whether that assembly is the only admissible one) remains open and is not addressed here.

OP-H status: CLOSED. Open Problem H is resolved. The formal derivation from P18’s $\hat{O}$ formalism, called for by P173 and P176, is complete.

References

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