

$G_1^{(171)}$ and the Near-Miss $\delta \approx h(G_2)$: A PSLQ Investigation of $j(i) - 4\Omega \approx 6$

Addendum 183 to the TOE Corpus

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

May 2026

Contents

1	Background and the two G_1 constants	2
1.1	Two constants sharing a name	2
1.2	Scope of this addendum	3
2	Exact values at 55 digits	3
3	PSLQ integer-relation search	4
3.1	Setup	4
3.2	Basis A: the trivial polynomial identity	4
3.3	Basis B: the residual $\delta - 6$ against CM/Gamma ingredients	4
3.4	Basis C: $G_1^{(171)}$ against a transcendental basis	4
3.5	Basis D: extended search including $e^{2\pi}$ and the Heegner number	4
3.6	Main theorem	5
4	Structural interpretation	5
4.1	Why the near-miss is a polynomial accident	5
4.2	The Coxeter-number interpretation is observational	5
4.3	Geometric reading via S^2	6
4.4	Contrast with P179	6
5	mpmath.identify probes	6
6	Summary and open directions	6
6.1	Main results	6
6.2	Open directions	7
7	Conclusion	7

Abstract

Addendum 171 (P171) introduced the constant

$$G_1^{(171)} = \frac{j(i)}{4\Omega} - 1 \approx 0.3456\%,$$

where $\Omega = \pi\alpha^{-1}$ is the TOE breath period ($\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$) and $j(i) = 1728$ is the Klein j -invariant at the CM point $\tau = i$. This is *not* the constant $G_1^{(179)} = 1728/\pi^3 - 1 \approx 54.73$ studied in Addendum 179 [2]; the two constants use different TOE denominators ($4\Omega \approx 1722$ versus $\pi^3 \approx 31.006$) and differ by a factor of roughly 158.

P171 proved that the absolute gap $\delta = j(i) - 4\Omega = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2 \approx 5.9510$ is close to but strictly less than $h(G_2) = J_{\text{short}}/2 = 6$ — the Coxeter number of G_2 and half the G_2 short-root sector sum $J_{\text{short}} = 12$. P171 left open the question of whether this near-miss has additional structure: is the residual $\delta - 6$ expressible in terms of CM periods, Gamma-function values, or other modular ingredients at $\tau = i$?

The present addendum provides a definitive PSLQ investigation of this question. We run four systematic integer-relation searches at `mp.dps = 55` over bases including $\Gamma(1/4)$, the lemniscate constant ϖ , $\log 2$, $\zeta(3)$, $e^{2\pi}$, and the Heegner-related quantity $e^{-\pi\sqrt{163}}$. All searches return **no relation** for $\delta - 6$ at maximal coefficient ≤ 1000 . The only integer relation found is the *trivial* polynomial identity $\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$, recovered when `maxcoeff` ≥ 1728 .

Our verdict: **the near-miss $\delta \approx 6 = h(G_2)$ is a polynomial coincidence**. The gap δ is entirely determined by its definition as a polynomial in π ; the proximity to the Coxeter number $h(G_2) = 6$ carries no additional CM, modular, or Gamma-function structure at coefficient norm ≤ 1000 . All numerical claims are verified to 55 significant figures by `verify_P183.py`.

1 Background and the two G_1 constants

1.1 Two constants sharing a name

The TOE corpus contains two constants that go by the name G_1 , arising from two distinct TOE denominators. It is essential to keep them separate.

Definition 1.1 (The two G_1 constants).

$$G_1^{(171)} = \frac{j(i)}{4\Omega} - 1, \quad \Omega = \pi\alpha^{-1}, \quad \alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \quad (1)$$

$$G_1^{(179)} = \frac{j(i)}{4\Omega_0} - 1, \quad \Omega_0 = \frac{\pi^3}{4}, \quad (2)$$

where $j(i) = 1728 = 12^3$ is the Klein j -invariant at $\tau = i$.

Assertion 1.1 (Numerical distinction). The two constants are numerically far apart:

$$4\Omega = 16\pi^4 + 4\pi^3 + 4\pi^2 \approx 1722.049, \quad (3)$$

$$4\Omega_0 = \pi^3 \approx 31.006, \quad (4)$$

giving $G_1^{(171)} \approx 0.003456$ (sub-percent) while $G_1^{(179)} \approx 54.73$ (large). Their ratio is $(1 + G_1^{(179)})/(1 + G_1^{(171)}) = 4\Omega/(pi^3) \approx 55.5$. The two constants are independent transcendentals.

Assertion 1.2 (Prior results in the corpus). P171 [1] proved:

(i) $G_1^{(171)}$ is transcendental.

(ii) $j(i) - 4\Omega = 5.9510\dots < 6 = h(G_2)$ (strictly non-exact).

(iii) $4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}$ (sphere-volume identity).

P179 [2] proved that $G_1^{(179)}$ lies outside the Chowla–Selberg $\mathbb{Q}$ -linear span at $\tau = i$ (PSLQ: no relation at coeff ≤ 1000). The present addendum extends the PSLQ investigation to $G_1^{(171)}$ and to the near-miss residual $\delta - h(G_2)$.

1.2 Scope of this addendum

P171 identified the near-miss $\delta = j(i) - 4\Omega \approx 5.951 \approx h(G_2) = 6$ as an open structural question. The proximity to the Coxeter number of G_2 is intriguing: $j(i) = J_{\text{short}}^3 = 12^3$ already involves the G_2 short-root sum $J_{\text{short}} = 12$, and $h(G_2) = J_{\text{short}}/2 = 6$ is the Coxeter number of the same Lie algebra. If $\delta = 6$ were exact, the identity $j(i) = 4\Omega + h(G_2)$ would express the j -invariant as the S^2 -volume-weighted coupling constant plus the G_2 Coxeter number — a beautiful TOE identity. P171 proved it is not exact. This addendum asks: is the near-miss *structured* in some weaker sense?

2 Exact values at 55 digits

Assertion 2.1 ($G_1^{(171)}$ at 55 significant figures). At `mp.dps = 55`:

$$\alpha^{-1} = 137.0363037758784325592023946515612348284120230701 \dots \quad (5)$$

$$\Omega = 430.5122452173989277400721368217979913364493306638 \dots \quad (6)$$

$$4\Omega = 1722.0489808695957109602885472871919653457973226555 \dots \quad (7)$$

$$G_1^{(171)} = 0.003455778085591479075380623365743477970730119293 \dots \quad (8)$$

Equivalently, $G_1^{(171)} \approx 0.34558\%$ of 4Ω . Verified by `verify_P183.py`.

Assertion 2.2 (The near-miss gap δ at 55 digits).

$$\delta = j(i) - 4\Omega = 5.9510191304042890397114527128080346542026773445 \dots \quad (9)$$

$$h(G_2) = 6 - \delta = 0.0489808695957109602885472871919653457973226554703 \dots \quad (10)$$

The deficit $6 - \delta > 0$ confirms P171's non-exactness. In percentage terms, $|\delta - 6|/|\delta| \approx 0.823\%$.

Assertion 2.3 (Exact polynomial form of δ). By the definition $\Omega = \pi\alpha^{-1} = 4\pi^4 + \pi^3 + \pi^2$:

$$\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2. \quad (11)$$

The deficit likewise has an exact polynomial form:

$$6 - \delta = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722 = 4\pi\alpha^{-1} - 1722. \quad (12)$$

Both are transcendental (nonconstant polynomials in π minus rationals).

Proof. Substituting $4\Omega = 16\pi^4 + 4\pi^3 + 4\pi^2$ into $\delta = 1728 - 4\Omega$ gives (11). Then $6 - \delta = 6 - (1728 - 16\pi^4 - 4\pi^3 - 4\pi^2) = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722$. Transcendence follows from Lindemann–Weierstrass applied to π . $\square$

Assertion 2.4 (Factored form of 4Ω). The quantity 4Ω admits the factored forms:

$$4\Omega = 4\pi \cdot \alpha^{-1} = 4\pi^2(4\pi^2 + \pi + 1) = \text{vol}(S^2) \cdot \alpha^{-1}. \quad (13)$$

In particular, $4\pi^2(4\pi^2 + \pi + 1) = 4\Omega$ identically (verified to 55 digits: difference is 0), confirming the factored form as an identity, not an approximation.

Proof. $4\Omega = 4\pi\alpha^{-1} = 4\pi(4\pi^3 + \pi^2 + \pi) = 16\pi^4 + 4\pi^3 + 4\pi^2 = 4\pi^2(4\pi^2 + \pi + 1)$. The factor $4\pi = \text{vol}(S^2)$ is the standard surface area of the unit 2-sphere. $\square$

3 PSLQ integer-relation search

3.1 Setup

The PSLQ algorithm [3] searches for integer coefficients $(c_1, \dots, c_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $\sum c_k x_k = 0$ for a given vector of real numbers $(x_1, \dots, x_n)$ computed to high precision. All runs use `mpmath.pslq` at `mp.dps = 55`, which provides approximately 50 guard digits beyond the working precision. The maximal coefficient bound is stated with each search.

We distinguish between the *trivial polynomial identity* (which any PSLQ must recover if `maxcoeff` is large enough) and *additional structure* (any relation not derivable from the definition of δ).

3.2 Basis A: the trivial polynomial identity

Assertion 3.1 (PSLQ Basis A — polynomial identity recovered). Over the basis $\{1, \delta, \pi^2, \pi^3, \pi^4\}$ with `maxcoeff` = 2000, PSLQ returns the unique relation

$$-1728 + \delta + 4\pi^2 + 4\pi^3 + 16\pi^4 = 0, \quad (14)$$

equivalently $\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$. Verification residual: $|-2.09 \times 10^{-53}|$ (numerical round-off only). Note: `maxcoeff` = 200 fails to recover this relation because the coefficient 1728 exceeds the search bound; the coefficient $1728 = j(i)$ is forced by the definition of δ .

This relation carries no information beyond the definition (11). The question is whether $\delta - 6$ or $G_1^{(171)}$ has *additional* structure beyond this polynomial form.

3.3 Basis B: the residual $\delta - 6$ against CM/Gamma ingredients

Assertion 3.2 (PSLQ Basis B — no CM/Gamma relation). Over the basis

$$\{\delta - 6, \pi, \pi^2, \pi^3, \pi^4, \Gamma(\tfrac{1}{4}), \log 2, \zeta(3)\}$$

with `maxcoeff` = 500, PSLQ returns **no relation**. If any $\mathbb{Q}$ -linear combination of these elements equals $\delta - 6$, all integer coefficients exceed 500 in absolute value.

3.4 Basis C: $G_1^{(171)}$ against a transcendental basis

Assertion 3.3 (PSLQ Basis C — no relation for $G_1^{(171)}$). Over the basis

$$\{G_1^{(171)}, \pi, \pi^2, \pi^3, \Gamma(\tfrac{1}{4}), \log 2, \zeta(3), \sin^4(\tfrac{\pi}{14})\}$$

(the last element drawn from the P175 δ_{QLC} basis) with `maxcoeff` = 500, PSLQ returns **no relation**. The constant $G_1^{(171)}$ has no expression over this basis at coefficient norm ≤ 500 .

3.5 Basis D: extended search including $e^{2\pi}$ and the Heegner number

Assertion 3.4 (PSLQ Basis D — no relation over extended basis). Over the ten-element basis

$$\{\delta - 6, \pi, \pi^2, \pi^3, \pi^4, \alpha^{-1}, \Gamma(\tfrac{1}{4}), \log 2, e^{2\pi}, e^{-\pi\sqrt{163}}\}$$

with `maxcoeff` = 1000, PSLQ returns **no relation** for $\delta - 6$. Remark: the element α^{-1} is included but PSLQ finds no relation for $\delta - 6$; the only linear relation among the basis elements themselves (with α^{-1} present) is the trivial $\alpha^{-1} - 4\pi^3 - \pi^2 - \pi = 0$, which PSLQ correctly recovers but which does not involve $\delta - 6$.

3.6 Main theorem

Theorem 3.1 (The near-miss $\delta \approx 6$ is a polynomial coincidence). *Let*

$$\mathcal{B} = \{1, \pi, \pi^2, \pi^3, \pi^4, \Gamma(\tfrac{1}{4}), \log 2, \zeta(3), e^{2\pi}, e^{-\pi\sqrt{163}}\}.$$

If $\delta - 6$ satisfies any $\mathbb{Q}$ -linear relation $\sum_{b \in \mathcal{B}} c_b b = c_0(\delta - 6)$ with $c_0, (c_b) \in \mathbb{Z}$, then

$$\|(c_0, (c_b)_{b \in \mathcal{B}})\|_\infty > 1000.$$

Consequently, the proximity of $\delta = j(i) - 4\Omega$ to $h(G_2) = 6$ is not explained by any combination of Chowla–Selberg periods, Gamma-function values, logarithms, $e^{2\pi}$, or the Ramanujan–Heegner exponential at coefficient norm ≤ 1000 . The only exact identity involving δ is the trivial polynomial $\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$.

Proof. Assertions 3.2–3.4 cover all four searches. Basis B covers $\Gamma(1/4)$, $\log 2$, $\zeta(3)$ at coeff ≤ 500 . Basis D extends to include $e^{2\pi}$ and $e^{-\pi\sqrt{163}}$ at coeff ≤ 1000 . Together, the searches span the Chowla–Selberg basis for $D = -4$ and key transcendentals associated with the CM point $\tau = i$. Since PSLQ is certified to find any relation at the stated coefficient bound given sufficient precision [3], and 55-digit precision is sufficient for coefficient bounds well below 10^{10} , the stated lower bound on the coefficient norm holds. $\square$

Corollary 3.2 ($\delta - 6$ lies outside the CM period world). *The residual $\delta - 6 = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722$ is not expressible as a rational-coefficient linear combination of the real period $\omega_E = \Gamma(1/4)^2/(4\sqrt{\pi})$ of the CM elliptic curve $E : y^2 = x^3 - x$, powers of π , logarithms, or $e^{2\pi}$ at coefficient norm ≤ 1000 .*

Proof. The CM period ω_E lies in the $\mathbb{Q}$ -linear span of $\Gamma(1/4)^2$ and $\pi^{1/2}$ by the Chowla–Selberg formula [4]. Both $\Gamma(1/4)$ and $\pi^{1/2}$ appear in Basis D. The claim follows from Theorem 3.1. $\square$

4 Structural interpretation

4.1 Why the near-miss is a polynomial accident

Assertion 4.1 (The deficit is itself a polynomial in π). The deficit

$$6 - \delta = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722 = 4\pi\alpha^{-1} - 1722 \quad (15)$$

is a nonconstant polynomial in π with integer coefficients, evaluated at $\pi \approx 3.14159$. Its value ≈ 0.04899 is an accident of the specific numerical value of π : there is no number-theoretic mechanism forcing this quantity to be small relative to $h(G_2) = 6$. Numerically, $|6 - \delta|/|6| \approx 0.82\%$, which while small is not exceptionally so by the standards of polynomial-in- π coincidences.

Remark 4.1. The deficit $4\pi\alpha^{-1} - 1722$ can be bounded by noting that $\alpha^{-1} \approx 137.036$ while $1722/(4\pi) \approx 136.992$. The difference $\alpha^{-1} - 1722/(4\pi) \approx 0.044$ is entirely determined by the specific coefficients in $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ and the integer 1722. There is no reason from Lie theory, CM theory, or modular arithmetic why this difference should be small; it merely is.

4.2 The Coxeter-number interpretation is observational

Assertion 4.2 (The Coxeter identity fails to be exact). The identity $j(i) = 4\Omega + h(G_2)$ — which would express the j -invariant at $\tau = i$ as the S^2 -weighted coupling constant plus the G_2 Coxeter number — is *not* exact. The precise statement (P171 [1]) is:

$$j(i) = 4\Omega + \delta, \quad \delta = 5.9510\dots \neq 6 = h(G_2).$$

The near-miss $\delta \approx h(G_2)$ is therefore an observation, not a theorem. The G_2 Coxeter number appears in the TOE through the root system and the short-root sector sum $J_{\text{short}} = 12$ (P151); its relationship to δ is numerical rather than algebraic.

Assertion 4.3 (No Coxeter mechanism connects δ and 6). The Lie-theoretic interpretation of $h(G_2) = 6$ is that G_2 has the Coxeter number $h = 2m + 2$ where $m = 2$ is the number of short positive roots; equivalently $h(G_2) = |\Phi|/\text{rank}(G_2) = 12/2 = 6$. This combinatorial quantity is an integer, while $\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$ is transcendental. No algebraic mechanism can make a transcendental polynomial in π equal to an integer, so the near-miss is irreducibly a coincidence.

4.3 Geometric reading via S^2

Assertion 4.4 (S^2 interpretation of the near-miss). Using $4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}$ (P171, Assertion 2.4 above), the near-miss reads:

$$j(i) \approx \text{vol}(S^2) \cdot \alpha^{-1} + h(G_2). \quad (16)$$

In words: the j -invariant at the CM point $\tau = i$ is *approximately* equal to the product of the S^2 surface area and the TOE fine-structure denominator, plus the G_2 Coxeter number. The approximation is off by ≈ 0.049 , roughly 0.003% of $j(i)$. This is a striking numerical coincidence but, by Theorem 3.1, not a structural identity.

4.4 Contrast with P179

Assertion 4.5 (Contrast with P179's negative PSLQ result). P179 [2] found that $G_1^{(179)} = 1728/\pi^3 - 1 \approx 54.73$ lies outside the Chowla–Selberg $\mathbb{Q}$ -linear span at coefficient norm ≤ 1000 . The present addendum finds the same conclusion for $G_1^{(171)} \approx 0.003456$ and for $\delta - 6$: neither has additional structure over the CM/Gamma/Heegner basis. Both constants sit at the interface between the algebraic CM world ($j(i) = 1728 \in \mathbb{Z}$) and the transcendental π -polynomial world (the TOE geometric constants). The Chowla–Selberg formula mediates between modular arithmetic and Gamma values but cannot reach purely polynomial expressions in π .

Remark 4.2 (Why the Chowla–Selberg formula cannot reach $\delta - 6$). The Chowla–Selberg formula for $D = -4$ expresses the real period $\omega_E = \Gamma(1/4)^2/(4\sqrt{\pi})$ of the CM elliptic curve $E : y^2 = x^3 - x$ in terms of $\Gamma(1/4)$ and $\pi^{1/2}$. The basis elements $\Gamma(1/4)^k$ and $\pi^{k/2}$ are periods or quasi-periods of E . In contrast, $\delta - 6 = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722$ involves π^2, π^3, π^4 but not $\pi^{1/2}$ or $\Gamma(1/4)$; the polynomial structure in integer powers of π is not accessible from the period world. By Nesterenko's theorem [7], π, e^π , and $\Gamma(1/4)$ are algebraically independent over $\mathbb{Q}$, which implies that no polynomial in π with rational coefficients can equal any element of the $\mathbb{Q}[\Gamma(1/4)]$ ring.

5 mpmath.identify probes

Assertion 5.1 (No closed form found by `identify`). The `mpmath.identify` function (tolerance 10^{-14} at 55 digits) returns no closed-form expression for any of the following quantities: $G_1^{(171)}$, δ , $\delta - 6$, $G_1^{(171)} \cdot \pi$, $G_1^{(171)} \cdot \pi^2$, δ/π , $(\delta - 6)/\pi^2$, $(\delta - 6)/\Gamma(1/4)$, and $6 \cdot G_1^{(171)}$. The only spurious matches (using low-denominator exponents over small integers) appear for $G_1^{(171)} \cdot \pi$ and $(\delta - 6) \cdot \alpha^{-1}$, which `identify` associates with expressions of the form (p/q) -rational exponents of 2, 3, 5, 7; these do not survive cross-checking and are artefacts of the numerical search algorithm.

6 Summary and open directions

6.1 Main results

The central result of this addendum is:

$$G_1^{(171)} = \frac{j(i)}{4\Omega} - 1 \approx 3.456 \times 10^{-3}, \quad \delta = j(i) - 4\Omega = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2 \approx 5.9510, \quad (17)$$

with the PSLQ verdict:

$$\text{VERDICT: COINCIDENCE. } \delta \approx 6 = h(G_2) \text{ is a polynomial accident in } \pi; \text{ no additional CM structure at } c \quad (18)$$

The supporting results are:

$$\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2 \quad (\text{only exact relation, Assertion 3.1}), \quad (19)$$

$$\delta - 6 \notin \mathbb{Q}\text{-span}(\{1, \pi, \pi^2, \pi^3, \pi^4, \Gamma(\tfrac{1}{4}), \log 2, \zeta(3), e^{2\pi}, e^{-\pi\sqrt{163}}\}), \quad \|\text{coeff}\|_\infty \leq 1000 \quad (20)$$

(Theorem 3.1),

$$j(i) = 4\Omega + \delta \approx \text{vol}(S^2) \cdot \alpha^{-1} + h(G_2) \quad (\text{near-identity, not exact, Assertion 4.4}). \quad (21)$$

6.2 Open directions

1. *Coefficient lower bound.* Theorem 3.1 gives $\|\text{coeff}\|_\infty > 1000$. Improving this to an analytic lower bound — e.g., by Gel'fond–Schneider or Baker-type results on linear forms in logarithms — would convert the computational result into a theorem. Nesterenko's algebraic independence of $\pi, e^\pi, \Gamma(1/4)$ implies that $\delta - 6$, being a polynomial in π , is algebraically independent of $\Gamma(1/4)$ and e^π over $\mathbb{Q}$, giving a partial step in this direction.
2. *The deficit* $6 - \delta \approx 0.04898$. This quantity equals $16\pi^4 + 4\pi^3 + 4\pi^2 - 1722 = 4\pi\alpha^{-1} - 1722$. P171 [1] remarked that $6 - \delta \approx \pi^2/200$ (with $\pi^2/200 \approx 0.04935$), a further near-miss within the near-miss. Whether this sub-near-miss has any combinatorial interpretation is open.
3. *Extending PSLQ to $\Gamma(1/3)$, $\Gamma(1/8)$, and other CM discriminants.* The present analysis is restricted to the $D = -4$ Chowla–Selberg basis. CM points at discriminants $D = -3, -7, -11, -163$ produce different Gamma-function ingredients. Whether $\delta - 6$ is reachable from any of these is not tested here.
4. *The Hopf fibration interpretation.* Assertion 4.4 expresses the near-miss as $j(i) \approx \text{vol}(S^2) \cdot \alpha^{-1} + h(G_2)$. The S^3 geometry carries a Hopf fibration with S^2 base. Whether the G_2 Coxeter number $h(G_2) = 6$ appears naturally as a degree or winding number in this fibration is an open geometric question.
5. *Relation to P179.* The ratio $(G_1^{(179)} + 1)/(G_1^{(171)} + 1) = 4\Omega/\pi^3 = \Omega/\Omega_0$ is the ratio of the two TOE periods. Whether the PSLQ negative results for $G_1^{(171)}$ and $G_1^{(179)}$ are related — e.g., whether one being outside the CM frame forces the other to be — is unknown.

7 Conclusion

The TOE near-miss $j(i) - 4\Omega \approx 6 = h(G_2)$ was identified in P171 [1] as a compelling but unproven structural observation. The present addendum closes the computational investigation: PSLQ over four independent bases covering the Chowla–Selberg ingredients for the CM elliptic curve $E : y^2 = x^3 - x$ at $\tau = i$, together with $\log 2, \zeta(3), e^{2\pi}$, and $e^{-\pi\sqrt{163}}$, returns no integer relation for the residual $\delta - 6$ at coefficient norm ≤ 1000 .

The verdict is unambiguous: **the near-miss $\delta \approx h(G_2)$ is a polynomial coincidence.** The gap $\delta = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$ is entirely defined by the polynomial structure of the TOE

breath period $\Omega = \pi\alpha^{-1}$; the proximity to the G_2 Coxeter number $h(G_2) = 6$ is a numerical accident. No Chowla–Selberg period, no Gamma-function value, and no modular quantity at $\tau = i$ explains the discrepancy from exactness.

This places $G_1^{(171)}$ alongside P179’s $G_1^{(179)}$ as a genuinely transcendental structural constant of the TOE that sits irreducibly outside the CM period world. Both constants arise from the ratio of the algebraic CM value $j(i) = 1728$ to a π -polynomial TOE quantity; in neither case does the Chowla–Selberg framework absorb the gap. The TOE thus exhibits a systematic transcendental interface at $\tau = i$: its geometric constants (Ω and Ω_0) approach the CM modular slot $j(i)$ closely but never reach it, and the shortfall is irreducible to any CM or Gamma-function structure at the tested coefficient scales.

References

- [1] L. F. Vlegels, *G_1 as a Modular Residual: The Fractional Gap between the TOE Breath Period and One Quarter of $j(i)$* , Addendum 171 to the TOE Corpus, May 2026.
- [2] L. F. Vlegels, *G_1 and the Chowla–Selberg Formula: Does the CM Period at $\tau = i$ Resolve the Modular Gap in $j(i)/(4\Omega_0)$?*, Addendum 179 to the TOE Corpus, May 2026.
- [3] H. R. P. Ferguson, D. H. Bailey and S. Arno, “Analysis of PSLQ, an integer relation finding algorithm,” *Math. Comp.* **68** (1999), 351–369.
- [4] S. Chowla and A. Selberg, “On Epstein’s zeta function (I),” *Proc. Nat. Acad. Sci. USA* **35** (1949), 371–374.
- [5] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.
- [6] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer, 1994.
- [7] Y. V. Nesterenko, “Modular functions and transcendence questions,” *Mat. Sb.* **187** (1996), 65–96.
- [8] S. Lang, *Elliptic Functions*, 2nd ed., Springer, 1987.
- [9] A. O. Gel’fond, *Transcendental and Algebraic Numbers*, Dover, 1960.