

Addendum 181 — $T_{2B}(\tau_{-163}) = -24$: Asymptotic Eta-Quotient Identity at the Ramanujan CM Point

Léon Fernando Vlegels

Independent Researcher

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Erratum (A343, 2026-06-15): The result — $T_{2B}(\tau_{-163}) \approx -24$ is a Ramanujan eta-expansion artefact, $-24 = -2J_{\text{short}} - \text{stands}$. Only the provenance “ J_{short} established in P163 ($T_{3B}(i) = 12$)” is corrected: per A337, $J_{\text{short}} = 12$ is established in P151/P158 as G_2 root geometry, and $j^{1/3}(i) = 12$ is the E8/G2 cube-root contact, *not* a Monster 3B McKay-Thompson value (canonical 3B = 535.59 at i). The T_{2B} scan and the -24 artefact flag are unaffected. See A337/A343. Body preserved as audit trail.

Abstract

Addendum P177 [3] scanned five McKay-Thompson-related series at all nine Stark–Heegner CM points and found four values in the TOE integer set S . One hit was flagged as structurally distinct: $T_{2B}(\tau_{-163}) = -24$, where $T_{2B}(\tau) := \text{Re}[(\eta(\tau/2)/\eta(\tau))^{24}]$ and $\tau_{-163} = (1 + i\sqrt{163})/2$ is the Ramanujan–Heegner point. No paper in the corpus explained why this coefficient lands in S .

The present addendum supplies the explanation. We prove the asymptotic identity

$$\text{Re} \left[\left(\frac{\eta(\tau_{-163}/2)}{\eta(\tau_{-163})} \right)^{24} \right] = -24 + 2048 e^{-\pi\sqrt{163}} + O(e^{-3\pi\sqrt{163}/2}),$$

with the leading correction $2048 e^{-\pi\sqrt{163}} \approx 7.80 \times 10^{-15}$ confirmed at 55 decimal digits (`verify_P181.py`). The coefficient $2048 = 2^{11} = \binom{24}{1} + \binom{24}{3}$ arises from the expansion of $\prod_{n \text{ odd}} (1 - p^n)^{24}$ at $p = e^{\pi i \tau_{-163}}$; it is not $24 e^{-\pi\sqrt{163}}$ as P177 Section 4.4 informally quoted.

We further show that $D = -163$ is the unique Stark–Heegner discriminant for which this correction falls below 10^{-10} : the threshold requires $|D| > 95.19$, and only $|D| = 163$ among the nine class-number-one discriminants satisfies this. The integer -24 belongs to S , equals $-2 \cdot J_{\text{short}} = -2 \times 12$, and enters S through the Leech-lattice dimension 24 (the exponent of the eta quotient). The identity is purely asymptotic; it is not an algebraic CM specialization.

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1 Introduction

1.1 Context

Addendum P177 [3] evaluated five series — T_{1A} , T_{2A} , T_{2B} , T_{3A} , T_{3B} — at the nine class-number-one CM points (the Stark–Heegner list [5]), yielding 45 evaluations at 55 decimal-digit precision. Four values landed in the TOE integer set

$$S = \{0, \pm 12, \pm 24, \pm 36, \pm 48, 72, 126, 240, 248, 744, 984, 1728, 196560, 196884\}. \quad (1)$$

Three of the four hits have clean algebraic explanations:

$$T_{3B}(\tau_i) = j(i)^{1/3} = 1728^{1/3} = 12 = J_{\text{short}} \quad (\text{algebraic CM identity, P163}), \quad (2)$$

$$T_{3B}(\tau_{-3}) = j(\tau_{-3})^{1/3} = 0^{1/3} = 0 \in S \quad (\text{trivial}), \quad (3)$$

$$T_{1A}(\tau_i) = j(i) - 744 = 1728 - 744 = 984 \in S \quad (\text{algebraic}). \quad (4)$$

The fourth hit, $(T_{2B}, D = -163) \mapsto -24$, was flagged in P177 as “structurally distinct from the T_{3B} hits — no paper explains why this coefficient lands in S .” It arises from an eta-quotient expansion, not from an algebraic CM value of any modular function. This addendum is the explanation.

1.2 Main findings

- Correct leading coefficient.** P177 Section 4.4 gave an informal derivation suggesting the leading correction to -24 was $24e^{-\pi\sqrt{163}}$. The correct coefficient is **2048**, not 24; see Assertion 8. The residual is 7.80×10^{-15} , not 9.14×10^{-17} .
- Exact asymptotic formula.** Assertion 7 gives the precise expansion $\text{Re}[(\eta(\tau/2)/\eta(\tau))^{24}]|_{\tau_{-163}} = -24 + 2048e^{-\pi\sqrt{163}} + O(e^{-3\pi\sqrt{163}/2})$.

3. **Uniqueness.** $D = -163$ is the unique Stark–Heegner discriminant for which $T_{2B}(\tau_D) \equiv -24 \pmod{10^{-10}}$; Assertion 9 derives the threshold $|D| > 95.19$.
4. $-24 \in S$. The value -24 is the universal leading real coefficient in the expansion, forced by the exponent 24 in $(\eta(\tau/2)/\eta(\tau))^{24}$. Its membership in S reflects the Leech-lattice dimension (Assertion 12).

2 The T_{2B} Function

2.1 Definition and notation

Let $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ denote the Dedekind eta function, $q = e^{2\pi i \tau}$, $\tau \in \mathfrak{H}$. Throughout this paper we set

$$p := e^{\pi i \tau},$$

the half-nome satisfying $p^2 = q$. The function studied in P177 is:

Assertion 1 (Definition of T_{2B}).

$$T_{2B}(\tau) := \operatorname{Re} \left[\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} \right].$$

We remark that T_{2B} as defined here is *not* the standard McKay–Thompson series for the Monster’s 2B conjugacy class in the sense of Conway–Norton [4]; the latter has integer-power q -expansion $q^{-1} + 276q + 2048q^2 + \dots$. Our T_{2B} has half-integer powers and is real-part-extracted. We retain P177’s naming convention throughout.

2.2 Product formula

Assertion 2 (Product formula).

$$\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} = p^{-1} \cdot \prod_{n=1,3,5,\dots}^{\infty} (1 - p^n)^{24}.$$

Proof. The eta function satisfies $\eta(\tau)^{24} = p^2 \prod_{n \geq 1} (1 - p^{2n})^{24}$ and $\eta(\tau/2)^{24} = p \prod_{n \geq 1} (1 - p^n)^{24}$. Therefore

$$\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} = \frac{p \prod_{n \geq 1} (1 - p^n)^{24}}{p^2 \prod_{n \geq 1} (1 - p^{2n})^{24}} = p^{-1} \cdot \frac{\prod_{n \geq 1} (1 - p^n)^{24}}{\prod_{n \geq 1} (1 - p^{2n})^{24}} = p^{-1} \prod_{n \text{ odd} \geq 1} (1 - p^n)^{24},$$

where the last equality uses $\prod (1 - p^n) / \prod (1 - p^{2n}) = \prod_{n \text{ odd}} (1 - p^n)$. $\square$

2.3 p -expansion

Assertion 3 (p -expansion). With $p = e^{\pi i \tau}$,

$$\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} = p^{-1} - 24 + 276p - 2048p^2 + 17250p^3 - \dots \quad (5)$$

Proof. From Assertion 2: $(\eta(\tau/2)/\eta(\tau))^{24} = p^{-1} \cdot P(p)^{24}$ where $P(p) = \prod_{n \text{ odd}} (1 - p^n)$. Expanding $P(p)^{24}$ to order p^3 :

$$\begin{aligned} (1 - p)^{24} &= 1 - 24p + 276p^2 - 2024p^3 + 10626p^4 - \dots, \\ (1 - p^3)^{24} &= 1 - 24p^3 + \dots \end{aligned}$$

The coefficient of p^3 in $P(p)^{24} = (1 - p)^{24} (1 - p^3)^{24} \dots$ is $-\binom{24}{3} - \binom{24}{1} = -2024 - 24 = -2048$. So

$$P(p)^{24} = 1 - 24p + 276p^2 - 2048p^3 + \dots,$$

and multiplying by p^{-1} gives the stated expansion. $\square$

2.4 Consistency with P163

Assertion 4 ($T_{2B}(i) = 8$). At $\tau = i$ (the CM point for $D = -4$), $T_{2B}(i) = 8$, consistent with Addendum P163.

Proof. From the CM identity $\eta(i/2)/\eta(i) = 2^{1/8}$ (P163 [1]): $(\eta(i/2)/\eta(i))^{24} = 2^{24/8} = 2^3 = 8$, real and positive. Hence $T_{2B}(i) = \text{Re}[8] = 8$. Verified at 55 digits in `verify_P181.py` with $|T_{2B}(i) - 8| < 10^{-45}$. $\square$

3 Nome Structure at τ_{-163}

3.1 The Ramanujan–Heegner point

The CM point $\tau_{-163} = (1 + i\sqrt{163})/2$ has $j(\tau_{-163}) = -640320^3$, the largest Heegner j -value. The famous identity [6] $e^{\pi\sqrt{163}} \approx 640320^3 + 744$ follows from the expansion of the j -function: $j(\tau) = q^{-1} + 744 + \dots$, so $e^{\pi\sqrt{163}} \approx |j(\tau_{-163})| + 744 = 640320^3 + 744$ to twelve decimal places.

Assertion 5 (Nome at τ_{-163}). At $\tau_{-163} = (1 + i\sqrt{163})/2$:

$$p = e^{\pi i \tau_{-163}} = e^{\pi i/2} \cdot e^{-\pi\sqrt{163}/2} = i \cdot e^{-\pi\sqrt{163}/2}, \quad (6)$$

$$p^{-1} = -i \cdot e^{\pi\sqrt{163}/2}, \quad (7)$$

$$p^2 = (i)^2 \cdot e^{-\pi\sqrt{163}} = -e^{-\pi\sqrt{163}}. \quad (8)$$

Numerically: $|p| = e^{-\pi\sqrt{163}/2} \approx 1.952 \times 10^{-9}$, and $e^{-\pi\sqrt{163}} \approx 3.809 \times 10^{-18}$.

Proof. Direct computation: $e^{2\pi i \tau_{-163}/2} = e^{\pi i(1+i\sqrt{163})/2} = e^{\pi i/2} \cdot e^{-\pi\sqrt{163}/2} = i \cdot e^{-\pi\sqrt{163}/2}$. $\square$

3.2 Imaginary part

Assertion 6 (Imaginary part is huge). $\text{Im}[(\eta(\tau_{-163}/2)/\eta(\tau_{-163}))^{24}] \approx -e^{\pi\sqrt{163}/2} \approx -5.124 \times 10^8$.

Proof. From Assertion 3, the leading imaginary contribution is $p^{-1} = -i \cdot e^{\pi\sqrt{163}/2}$, giving $\text{Im} \approx -e^{\pi\sqrt{163}/2}$. All other imaginary terms are $O(e^{-\pi\sqrt{163}/2})$, negligible at this scale. Numerically: $\text{Im} = -512384047.996\dots$, $-e^{\pi\sqrt{163}/2} = -512384048.000\dots$, relative error $\approx 10^{-15}$. $\square$

4 The Asymptotic Expansion at τ_{-163}

4.1 Separating real and imaginary parts

At τ_{-163} , the nome $p = i \cdot e^{-\pi\sqrt{163}/2}$ has phase $\pi/2$. Therefore powers of p alternate between real ($p^{2k} = (-1)^k e^{-k\pi\sqrt{163}}$) and imaginary ($p^{2k+1} = (-1)^k \cdot i \cdot e^{-(2k+1)\pi\sqrt{163}/2}$). Taking the real part of the expansion from Assertion 3:

Term	Value	Type	Magnitude
p^{-1}	$-i \cdot e^{\pi\sqrt{163}/2}$	imaginary	$\approx 5.12 \times 10^8$
-24	-24	real	24
$+276p$	$276i \cdot e^{-\pi\sqrt{163}/2}$	imaginary	$\approx 5.38 \times 10^{-7}$
$-2048p^2$	$+2048 \cdot e^{-\pi\sqrt{163}}$	real	$\approx 7.80 \times 10^{-15}$
$+17250p^3$	imaginary	imaginary	$\approx 3.31 \times 10^{-23}$
$\vdots$			

The real part is $-24 + 2048 e^{-\pi\sqrt{163}} + O(e^{-2\pi\sqrt{163}})$.

4.2 Main theorem

Assertion 7 (Asymptotic identity).

$$T_{2B}(\tau_{-163}) = \operatorname{Re} \left[\left(\frac{\eta(\tau_{-163}/2)}{\eta(\tau_{-163})} \right)^{24} \right] = -24 + 2048 e^{-\pi\sqrt{163}} + O(e^{-2\pi\sqrt{163}}). \quad (9)$$

Numerically: $2048 e^{-\pi\sqrt{163}} \approx 7.801 \times 10^{-15}$.

Proof. From Assertions 2–5 and the table above:

$$\begin{aligned} \left(\frac{\eta(\tau_{-163}/2)}{\eta(\tau_{-163})} \right)^{24} &= p^{-1} \cdot P(p)^{24} \Big|_{p=i \cdot e^{-\pi\sqrt{163}/2}} \\ &= p^{-1} (1 - 24p + 276p^2 - 2048p^3 + \dots) \\ &= \underbrace{p^{-1}}_{\text{imag}} - 24 + \underbrace{276p}_{\text{imag}} + 2048 e^{-\pi\sqrt{163}} + O(e^{-3\pi\sqrt{163}/2}). \end{aligned}$$

The real part is $-24 + 2048 e^{-\pi\sqrt{163}} + O(e^{-2\pi\sqrt{163}})$. Note: the $O(e^{-2\pi\sqrt{163}})$ term comes from p^4 (via $-2048p^2 \cdot (-24p^2)$ cross term $= +17250 \cdot \operatorname{Re}(p^4) = 17250 e^{-2\pi\sqrt{163}} \approx 3.9 \times 10^{-34}$). $\square$

4.3 The correction coefficient 2048

Assertion 8 (Correction coefficient). The leading correction to $T_{2B}(\tau_{-163}) = -24$ is $+2048 e^{-\pi\sqrt{163}}$ with $2048 = 2^{11} = \binom{24}{1} + \binom{24}{3}$.

Proof. From the proof of Assertion 3, the coefficient of p^3 in $P(p)^{24}$ is

$$-\binom{24}{3} - \binom{24}{1} = -2024 - 24 = -2048.$$

Here $-\binom{24}{3}$ arises from the p^3 term of $(1-p)^{24}$, and $-\binom{24}{1} = -24$ from the p^3 term of $(1-p^3)^{24}$ (which contributes $-24p^3$ to the product). Multiplying by p^{-1} : $p^{-1} \cdot (-2048p^3) = -2048p^2 = -2048 \cdot (-e^{-\pi\sqrt{163}}) = +2048 e^{-\pi\sqrt{163}}$.

Numerical verification: $|\text{residual}|/e^{-\pi\sqrt{163}} = 2048.000000$ to six significant figures (`verify_P181.py A5`). $\square$

Correction to P177 Section 4.4. P177 Section 4.4 derived the expansion by truncating at the p^1 term of $(1-p)^{24}$, giving $\eta(\tau/2)^{24} \approx p - 24p^2$ and quoting $24 e^{-\pi\sqrt{163}}$ as the leading correction. This expansion was applied to $\eta(\tau/2)^{24}$ and $\eta(\tau)^{24}$ separately, but the ratio introduces a cancellation that shifts the leading coefficient. The correct coefficient in the *ratio* $(\eta(\tau/2)/\eta(\tau))^{24}$ is 2048, not 24. The numerical result of P177 — that $T_{2B}(\tau_{-163})$ rounds to -24 within 10^{-10} — remains correct (since $7.80 \times 10^{-15} < 10^{-10}$); only the informal analytical explanation of the residual magnitude was imprecise.

5 Uniqueness Among Stark–Heegner Discriminants

5.1 Universal correction formula

The asymptotic expansion of Assertion 7 applies at every Stark–Heegner point $\tau_D = (b_D + i\sqrt{|D|})/2$ with $|D| > 3$. Writing $p_D = e^{\pi i \tau_D}$, the same argument gives:

$$T_{2B}(\tau_D) = -24 + 2048 e^{-\pi\sqrt{|D|}} + O(e^{-3\pi\sqrt{|D|}/2}). \quad (10)$$

The coefficient 2048 is *universal across all nine Stark–Heegner discriminants*: `verify_P181.py` confirms $|\text{residual}_D|/e^{-\pi\sqrt{|D|}} = 2048.00$ to five digits at $D \in \{-43, -67, -163\}$ (the three with $|D|$ large enough to suppress higher corrections). The coefficient at smaller $|D|$ is also close to 2048 ($D = -11$: 2046.5; $D = -19$: 2047.9).

Assertion 9 (Uniqueness threshold). The Stark–Heegner discriminant D satisfies $T_{2B}(\tau_D) \in [-24 - \varepsilon, -24 + \varepsilon]$ with $\varepsilon = 10^{-10}$ if and only if $2048 e^{-\pi\sqrt{|D|}} < 10^{-10}$, i.e.,

$$\pi\sqrt{|D|} > \ln(2048 \times 10^{10}) = \ln(2.048 \times 10^{13}) \approx 30.65,$$

equivalently $|D| > (30.65/\pi)^2 \approx 95.19$. Among the nine Stark–Heegner discriminants $|D| \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$, only $|D| = 163$ satisfies this. Hence $D = -163$ is the unique element of the Stark–Heegner list for which $T_{2B}(\tau_D)$ hits -24 at the 10^{-10} level.

Proof. The bound $|D| > 95.19$ follows algebraically. The table below verifies it numerically at the three closest candidates:

D	$e^{-\pi\sqrt{ D }}$	$ \text{residual} $	Hit?
-43	$\approx 1.13 \times 10^{-9}$	$\approx 2.32 \times 10^{-6}$	no
-67	$\approx 6.79 \times 10^{-12}$	$\approx 1.39 \times 10^{-8}$	no
-163	$\approx 3.81 \times 10^{-18}$	$\approx 7.80 \times 10^{-15}$	yes

The residual at $D = -67$ is $\approx 1.39 \times 10^{-8} > 10^{-10}$ (fails). The residual at $D = -163$ is $\approx 7.80 \times 10^{-15} < 10^{-10}$ (passes). No D with $|D| < 67$ can pass, since $e^{-\pi\sqrt{43}} \approx 1.13 \times 10^{-9}$ gives a residual $\approx 2.32 \times 10^{-6} \gg 10^{-10}$. $\square$

Assertion 10 ($D = -67$: near-miss). At $D = -67$, $T_{2B}(\tau_{-67}) = -24 + 1.39 \times 10^{-8} > -24 - 10^{-10}$, so τ_{-67} fails the hit criterion. The residual $/e^{-\pi\sqrt{67}} = 2048.00$ (five-digit agreement), confirming the universal coefficient.

Assertion 11 ($D = -43$: comfortable miss). At $D = -43$, $|T_{2B}(\tau_{-43}) - (-24)| \approx 2.31 \times 10^{-6} \gg 10^{-10}$. The residual $/e^{-\pi\sqrt{43}} = 2048.00$.

6 Connection to S and TOE Geometry

6.1 -24 as a TOE constant

Assertion 12 ($-24 \in S$). $-24 \in S$. In the TOE set S , the element -24 appears as

$$-24 = -2 \cdot 12 = -2 \cdot J_{\text{short}},$$

where $J_{\text{short}} = 12$ is the “short j ” constant established in P163 ($T_{3B}(i) = 12$). The signed pair ± 24 is a natural unit of the set S , which contains ± 24 and more generally $\{0, \pm 12k : 1 \leq k \leq 4\} \cup \{72, \dots\}$.

6.2 Why -24 and not another integer

Assertion 13 (The -24 is forced by the exponent). The coefficient -24 in $T_{2B}(\tau_{-163}) \approx -24$ is structurally forced: it is the coefficient of p^0 after multiplying p^{-1} by the leading p^1 term of $P(p)^{24} = (1-p)^{24} \dots$. Specifically, $p^{-1} \cdot (-\binom{24}{1} p) = -24$. Were the exponent in $(\eta(\tau/2)/\eta(\tau))^N$ equal to N (instead of 24), the asymptotic value at τ_{-163} would be $-N$, not -24 . The value -24 thus traces directly to the dimension of the Leech lattice ($N = 24$), which is also the exponent of the eta function $\eta^{24} = q \prod (1 - q^n)^{24}$ throughout Monstrous Moonshine.

6.3 Imaginary part and the Ramanujan near-integer

Assertion 14 (Imaginary part and Ramanujan). The imaginary part of $(\eta(\tau_{-163}/2)/\eta(\tau_{-163}))^{24}$ is approximately $-e^{\pi\sqrt{163}/2} \approx -5.12 \times 10^8$. This is not directly related to Ramanujan’s famous near-integer $e^{\pi\sqrt{163}} \approx 640320^3 + 744$, which arises from $|q|^{-1} = e^{\pi\sqrt{163}}$ in the j -function expansion. The two phenomena share the discriminant $D = -163$ and the exponential $e^{\pi\sqrt{163}}$ but are logically independent: Ramanujan’s identity is about T_{1A} , while the present paper concerns T_{2B} .

6.4 Asymptotic vs. algebraic hits

Assertion 15 (Asymptotic, not algebraic). The identity $T_{2B}(\tau_{-163}) \approx -24$ is *asymptotic*, not algebraic. The exact value is $T_{2B}(\tau_{-163}) = -24 + 7.800792959 \dots \times 10^{-15}$; it is irrational (it contains $e^{-\pi\sqrt{163}}$ transcendently). This contrasts with the algebraic hits: $T_{3B}(\tau_D) = j(\tau_D)^{1/3} \in \mathbb{Z}$ exactly for all nine Stark–Heegner discriminants (Lemma 3.2 of P177). The T_{2B} hit is therefore qualitatively different: it belongs to a class of *ε -integer asymptotic values* rather than exact CM specializations.

6.5 Full scan context

Assertion 16 (P177 scan summary). Among all 45 evaluations of $(T_{1A}, T_{2A}, T_{2B}, T_{3A}, T_{3B})$ at the nine Stark–Heegner CM points:

1. $T_{3B}(\tau_i) = 12 \in S \cap \{12^k\}$ — the unique 12^k hit (algebraic).
2. $T_{3B}(\tau_{-3}) = 0 \in S$ (trivial).
3. $T_{1A}(\tau_i) = 984 \in S$ (algebraic).
4. $T_{2B}(\tau_{-163}) = -24 \in S$ (asymptotic, $\varepsilon = 7.8 \times 10^{-15}$).

The T_{2B} hit is the deepest: its residual 7.8×10^{-15} is thirteen orders of magnitude below the hit threshold 10^{-10} , whereas the threshold discriminant for D-67 gives a residual only 1.4×10^{-8} , two orders above threshold.

7 T_{2B} at All Nine Stark–Heegner Points

Table 1 gives $T_{2B}(\tau_D)$ at all nine discriminants to 30 significant figures (extracted from `verify_P181.py`).

Table 1: $T_{2B}(\tau_D) = \text{Re}[(\eta(\tau_D/2)/\eta(\tau_D))^{24}]$ at all nine Stark–Heegner CM points. $\star$ marks the hit against S .

D	τ_D	$T_{2B}(\tau_D)$ (30 digits)	residual
-3	$(1 + i\sqrt{3})/2$	-16.0	8.00×10^0
-4	i	8.0	3.20×10^1
-7	$(1 + i\sqrt{7})/2$	-23.5	5.00×10^{-1}
-8	$i\sqrt{2}$	64.0	8.80×10^1
-11	$(1 + i\sqrt{11})/2$	-23.9389206454...	6.11×10^{-2}
-19	$(1 + i\sqrt{19})/2$	-23.9976871922...	2.31×10^{-3}
-43	$(1 + i\sqrt{43})/2$	-23.9999976851...	2.32×10^{-6}
-67	$(1 + i\sqrt{67})/2$	-23.9999999860...	1.39×10^{-8}
-163	$(1 + i\sqrt{163})/2$	-23.999999999999921... $\star$	7.80×10^{-15}

The convergence from above toward -24 is monotone for $|D| \geq 7$, driven by the exponential decrease of $2048 e^{-\pi\sqrt{|D|}}$. Note that $D \in \{-3, -4, -8\}$ give values far from -24 ; these arise

from CM identities involving small nomes where the p^{-1} and p terms contribute non-negligible real parts.

8 Conclusion

We have established the following chain of facts:

Main result. $T_{2B}(\tau_{-163}) = \operatorname{Re}[(\eta(\tau_{-163}/2)/\eta(\tau_{-163}))^{24}] = -24 + 2048 e^{-\pi\sqrt{163}} + O(e^{-2\pi\sqrt{163}})$, with residual 7.80×10^{-15} , verified numerically to 55 decimal digits (11 assertions, all passing).

Correction to P177. The correction coefficient is $2048 = 2^{11}$, not 24 as informally quoted in P177 Section 4.4. The factor $2048 = \binom{24}{1} + \binom{24}{3}$ arises from the p^3 coefficient of $\prod_{n \text{ odd}} (1 - p^n)^{24}$, where p is the half-nome at τ_{-163} .

Uniqueness. Among the nine Stark–Heegner discriminants, $D = -163$ is the unique one satisfying $|T_{2B}(\tau_D) - (-24)| < 10^{-10}$. The threshold $|D| > 95.19$ is crossed only at $|D| = 163$.

$-24 \in S$. The asymptotic value $-24 = -2 \cdot J_{\text{short}}$ belongs to the TOE set S . Its appearance traces to the exponent 24 (the Leech-lattice dimension) in the eta quotient: any function $(\eta(\tau/2)/\eta(\tau))^N$ would give $\approx -N$ at τ_{-163} .

Structural distinction. The T_{2B} hit is qualitatively different from the T_{3B} hits. The latter are exact algebraic integers (CM specializations of $j^{1/3}$); the former is an ε -integer, irrational, transcendental, and unique to $D = -163$ by an exponential hierarchy argument. Both types of hits deserve independent tracking in the 194-class programme of P170.

Open question. Is -24 the unique element of $S \cap (-\infty, 0)$ achievable as an asymptotic real-part value of an eta-quotient at a Stark–Heegner point? The answer depends on a systematic survey of eta quotients $(\eta(a\tau)/\eta(b\tau))^N$ at CM points, which is left to future work.

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