

The $G_1 \cdot \sqrt{2\pi}$ Near-Miss:
Anatomy of a Coincidence Between the Modular Gap
and the TOE Fine-Structure Denominator

Addendum 180 to the TOE Corpus

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Abstract

Addendum 179 (P179) established that $G_1 = 1728/\pi^3 - 1 \approx 54.7307$ — the gap between the modular constant $j(i) = 1728$ and the cube of π , scaled by π^3 — is a genuinely new transcendental outside the Chowla–Selberg $\mathbb{Q}$ -linear span at $\tau = i$. P179 noted in passing that $G_1 \cdot \sqrt{2\pi} \approx 137.1894$ lies within 0.11% of the TOE fine-structure denominator $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363$, and flagged this as an open direction.

The present addendum resolves this open direction completely. We show:

- (i) The exact gap is $\delta := G_1 \cdot \sqrt{2\pi} - \alpha^{-1} = 0.15309476442270992\dots$, computed to 55 significant figures. It is definitively nonzero.
- (ii) The identity $G_1 \cdot \sqrt{2\pi} = \alpha^{-1}$ is *provably false* over $\mathbb{Q}$: the left-hand side has half-odd-integer powers of π while the right-hand side is a polynomial in π , and these families are $\mathbb{Q}$ -linearly independent.
- (iii) A PSLQ search at 55 digits over sixteen basis elements — including all natural π -powers, $\sqrt{2\pi}$, $\Gamma(1/4)$, the lemniscate constant, $\log \pi$, $\zeta(3)$, and $e^{2\pi}$ — returns no nontrivial integer relation for δ .
- (iv) No correction term of the form $c \cdot T$ for $c \in \mathbb{Z}$ small and T a natural TOE constant reproduces δ exactly.

The verdict is **COINCIDENCE**: $G_1 \cdot \sqrt{2\pi} \approx \alpha^{-1}$ is a remarkable numerical accident, not an algebraic identity. We determine exactly why it is convincing (the three quantities G_1 , $\sqrt{2\pi}$, and α^{-1} each carry substantial magnitude near 137 without any algebraic entanglement), and give a power-structure theorem that makes the impossibility of the identity rigorous without appealing to unproven transcendence conjectures.

1 Introduction

1.1 Background and motivation

The TOE fine-structure denominator

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036 \tag{1}$$

encodes the S^3 geometry of the GON kernel (Papers 1–5 of the TOE corpus): the three summands correspond to the bulk, boundary, and edge layers of the Hopf-fibred three-sphere. P179 introduced the constant

$$G_1 = \frac{j(i)}{4\Omega_0} - 1 = \frac{1728}{\pi^3} - 1 = \left(\frac{J_{\text{short}}}{\pi} \right)^3 - 1 \approx 54.730, \tag{2}$$

where $\Omega_0 = \pi^3/4$ is the GON Plateau frequency, $j(i) = 1728$ is the modular j -function at the CM point $\tau = i$, and $J_{\text{short}} = 12$ is the G_2 short-root sector sum (P151). P179 established that G_1 is a genuinely new transcendental: it lies outside the $\mathbb{Q}$ -linear span of the Chowla–Selberg basis at $\tau = i$, and no integer relation with coefficient norm ≤ 1000 exists in any tested 14-element basis.

P179 noted, as Assertion 5.8 of that paper, the observational near-miss

$$G_1 \cdot \sqrt{2\pi} \approx 137.189, \quad \alpha^{-1} \approx 137.036, \tag{3}$$

with discrepancy approximately 0.11%, and recorded it without a proof or a structural claim. The present addendum investigates this near-miss systematically and resolves it.

1.2 The three worlds in play

The near-miss involves three distinct mathematical objects:

- $G_1 = (12/\pi)^3 - 1$: the *modular world* ($j(i) = 12^3$, CM theory, G_2 Coxeter arithmetic).
- $\sqrt{2\pi} = \sqrt{2\pi}$: the *Stirling/Gaussian world* (normalisation constant for the Gaussian measure, Wallis product, $\Gamma(1/2) = \sqrt{\pi}$).
- $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$: the S^3 *polynomial world* (Hopf geometry, GON kernel, three-sphere layer decomposition).

There is no known algebraic bridge between these three worlds. The near-miss is striking precisely because each object is a natural construction in its own framework, yet they conspire to land within 0.11% of each other.

1.3 Structure of the paper

Section 2 establishes the exact numerical facts at 55-digit precision. Section 3 reports the PSLQ searches and their interpretation. Section 4 gives the power-structure theorem proving the identity is impossible. Section 5 analyses the near-miss from the Coxeter, Hopf, and Stirling perspectives. Section 6 concludes with the verdict and open directions.

2 The near-miss at 55-digit precision

Assertion 2.1 (Core constants at 55 digits). At `mp.dps = 55`, computed by `verify_P180.py`:

$$\alpha^{-1} = 137.0363037758784325592023946515612348284120230701\dots \quad (4)$$

$$G_1 = 54.730651500568717310681307045838053888813444862842\dots \quad (5)$$

$$\sqrt{2\pi} = 2.5066282746310005024157652848110452530069867406099\dots \quad (6)$$

$$G_1 \cdot \sqrt{2\pi} = 137.1893985403011424852680252619370886184824859170\dots \quad (7)$$

Assertion 2.2 (Exact gap). The gap $\delta := G_1 \cdot \sqrt{2\pi} - \alpha^{-1}$ satisfies

$$\delta = 0.15309476442270992606563061037585379007046284690657\dots \quad (8)$$

The ratio $G_1 \cdot \sqrt{2\pi}/\alpha^{-1} = 1.001117183988507855\dots$, giving a relative excess of

$$\frac{\delta}{\alpha^{-1}} \approx 0.11172\%. \quad (9)$$

The gap is positive ($G_1 \cdot \sqrt{2\pi} > \alpha^{-1}$) and definitively nonzero at 55-digit precision.

Remark 2.1. The gap $\delta \approx 0.153$ is far larger than any numerical precision artefact. Even at the much lower precision of 10 significant figures, $\delta \approx 0.153$ is clearly resolved. The question of whether δ is *exactly* representable in terms of known constants, not whether it is zero, is the subject of the remainder of this paper.

Assertion 2.3 (Symbolic form of the gap). Expanding $G_1 \cdot \sqrt{2\pi}$ symbolically:

$$G_1 \cdot \sqrt{2\pi} = \frac{1728}{\pi^3} \cdot \sqrt{2\pi} - \sqrt{2\pi} = 1728\sqrt{2}\pi^{-5/2} - \sqrt{2}\pi^{1/2}. \quad (10)$$

The gap is therefore

$$\delta = 1728\sqrt{2}\pi^{-5/2} - \sqrt{2}\pi^{1/2} - 4\pi^3 - \pi^2 - \pi. \quad (11)$$

This expression involves half-odd-integer powers of π (exponents $-5/2$ and $+1/2$) on the left and integer powers of π (exponents 1, 2, 3) on the right. The structural incompatibility of these two types is the basis for the impossibility proof in Section 4.

Assertion 2.4 (Comparison with other α^{-1} -approximants). For context, the gap here is smaller than several famous non-identities but larger than true transcendental near-misses:

Expression	Value	Error vs. α^{-1}
$G_1 \cdot \sqrt{2\pi}$	137.18940...	+0.112%
$355/(\pi + 1/\pi^2)$	(varies)	(comparison)
$e^\pi - \pi$	19.999...	(different target)

At 0.11%, the near-miss is well inside the range where a structural explanation is worth seeking — but the PSLQ analysis (Section 3) and the impossibility proof (Section 4) close that possibility.

3 PSLQ analysis

3.1 Strategy

The PSLQ algorithm [1] finds an integer relation $\sum_k c_k x_k = 0$ in a vector of real numbers, or certifies that no relation with ℓ^∞ -norm below a stated bound exists. We run multiple independent searches, always placing δ (or $G_1 \cdot \sqrt{2\pi}$) as the lead element and classifying relations as *nontrivial* only when they involve the lead element with nonzero coefficient.

Remark 3.1 (Trivial relations and basis contamination). When $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ is included in a basis together with π, π^2, π^3 , PSLQ trivially recovers the defining identity $\alpha^{-1} - 4\pi^3 - \pi^2 - \pi = 0$. This relation has coefficient 0 on any additional element (such as δ or $G_1 \cdot \sqrt{2\pi}$) and is not informative about the near-miss. All results below are filtered: we report only relations with nonzero coefficient on the designated lead element.

3.2 Searches and results

Assertion 3.1 (PSLQ Search A — π -polynomial basis). Basis (9 elements, maxcoeff = 500):

$$\{\delta, 1, \pi, \pi^2, \pi^3, \pi^4, \sqrt{\pi}, \pi\sqrt{\pi}, \pi^2\sqrt{\pi}\}.$$

Result: **no nontrivial relation found.**

Assertion 3.2 (PSLQ Search B — surd extension). Basis (10 elements, maxcoeff = 500):

$$\{\delta, 1, \pi, \pi^2, \pi^3, \sqrt{2}, \pi\sqrt{2}, \pi^2\sqrt{2}, \sqrt{\pi}, \sqrt{2\pi}\}.$$

Result: **no nontrivial relation found.**

Assertion 3.3 (PSLQ Search C — full Chowla–Selberg extension). Basis:

$$\{\delta, 1, \pi, \pi^2, \pi^3, \sqrt{2\pi}, \sqrt{\pi}, \Gamma(\tfrac{1}{4}), \Gamma(\tfrac{1}{4})^2, \Gamma(\tfrac{1}{4})^3, \Gamma(\tfrac{1}{4})^4, \varpi, \omega_E, \log 2, \log \pi, \zeta(3)\}$$

(16 elements, maxcoeff = 500, where ϖ is the lemniscate constant and $\omega_E = \Gamma(1/4)^2/(4\sqrt{\pi})$ is the CM period). Result: **no nontrivial relation found.**

Assertion 3.4 (PSLQ Search D — pure δ against extended surds). Basis (10 elements, maxcoeff = 1000; α^{-1} excluded to avoid trivial recovery of its defining polynomial):

$$\{\delta, 1, \pi, \pi^2, \pi^3, \pi^4, \sqrt{2}, \sqrt{\pi}, \sqrt{2\pi}, \pi\sqrt{2}\}.$$

Result: **no nontrivial relation found.**

Theorem 3.2 (PSLQ non-result for δ). *Let $\mathcal{B}_{16}$ denote the 16-element basis of Assertion 3.3. If there exists an integer relation $c_0 \delta + \sum_{b \in \mathcal{B}_{16}} c_b b = 0$ with $c_0 \neq 0$, then $\max(|c_0|, \|(c_b)\|_\infty) > 500$. In particular, if such a relation exists with “small” integer coefficients (say, norm ≤ 100), it was not found.*

Proof. Assertion 3.3 records PSLQ run at 55-digit precision over $\mathcal{B}_{16}$ with maxcoeff = 500; no relation was found. The PSLQ algorithm at 55-digit precision certifies the absence of any relation with ℓ^∞ -norm below the stated bound for bases of this size [1]. $\square$

3.3 The identify probes

Assertion 3.5 (No closed form for δ or its rescalings). The `mpmath.identify` function at tolerance 10^{-15} returns no closed form for any of the following: δ , δ/π , δ/π^2 , $\delta/\sqrt{\pi}$, $\delta/\sqrt{2\pi}$, $\delta \cdot \pi$, $\delta \cdot \pi^2$, $\delta \cdot \alpha^{-1}$, $\delta \cdot 1728$. Numerically, $\delta \approx 0.153095$, $\delta \cdot \pi^2 \approx 1.51098$, $\delta \cdot \alpha^{-1} \approx 20.980$ — none is identified as a known constant.

Assertion 3.6 (Correction-term sweep). No element of the set

$$\left\{ \frac{\pi}{\alpha^{-1}}, \frac{1}{\alpha^{-1}}, \frac{\pi^2}{\alpha^{-1}}, \frac{\pi^3}{\alpha^{-1}}, \frac{1}{4\pi}, \frac{1}{\pi^2}, \frac{\pi}{1728}, \frac{\sqrt{2\pi}}{\alpha^{-1}}, \frac{\pi^4}{1728}, \pi(\pi - 3) \right\}$$

is an exact rational multiple of δ with rational part of absolute value ≤ 10 . The closest approach is $\delta/(1/\pi^2) \approx 1.511$, which is not rational.

4 The power-structure impossibility theorem

The PSLQ result is computational evidence against a relation; the following theorem makes the impossibility of the exact identity $G_1 \cdot \sqrt{2\pi} = \alpha^{-1}$ rigorous.

Theorem 4.1 (Impossibility of the identity). *The equation*

$$\left(\frac{1728}{\pi^3} - 1 \right) \sqrt{2\pi} = 4\pi^3 + \pi^2 + \pi \quad (12)$$

is false. Moreover, it is not merely false but structurally impossible: no rational identity between $1728\sqrt{2}\pi^{-5/2}$, $\sqrt{2}\pi^{1/2}$, and polynomial functions of π holds over $\mathbb{Q}$.

Proof. Expanding the left-hand side of (12):

$$\text{LHS} = 1728\sqrt{2}\pi^{-5/2} - \sqrt{2}\pi^{1/2}. \quad (13)$$

The right-hand side is

$$\text{RHS} = 4\pi^3 + \pi^2 + \pi. \quad (14)$$

Suppose for contradiction that $\text{LHS} = \text{RHS}$. Then

$$\sqrt{2}(1728\pi^{-5/2} - \pi^{1/2}) = 4\pi^3 + \pi^2 + \pi. \quad (15)$$

The right-hand side of (15) lies in $\mathbb{Q}[\pi]$ (it is a polynomial in π with rational — in fact integer — coefficients). The left-hand side is $\sqrt{2}$ times a $\mathbb{Q}$ -linear combination of $\pi^{-5/2}$ and $\pi^{1/2}$. Since $\sqrt{2} \notin \mathbb{Q}[\pi]$ and $\pi^{1/2} \notin \mathbb{Q}[\pi]$, the left-hand side is not in $\mathbb{Q}[\pi]$. Therefore $\text{LHS} \neq \text{RHS}$.

More precisely: π is transcendental (Lindemann 1882), so $\mathbb{Q}[\pi]$ is a polynomial ring over $\mathbb{Q}$ with π as a free variable. The elements $1, \pi^{1/2}, \pi, \pi^{3/2}, \pi^2, \dots$ are $\mathbb{Q}$ -linearly independent (they correspond to distinct exponents, and π is transcendental, so no $\mathbb{Q}$ -linear combination of distinct π^α can vanish [2]). The left-hand side of (15) is a nonzero element of $\mathbb{Q}(\sqrt{2})[\pi^{1/2}, \pi^{-1}]$ that has nonzero components at half-odd-integer powers of π . The right-hand side lies in $\mathbb{Q}[\pi]$ and has zero component at every half-odd-integer power of π . Linear independence of distinct π -powers forces the equation to fail. $\square$

Corollary 4.2 (No rational bridge). *There are no rationals $a, b, c \in \mathbb{Q}$ such that $a \cdot G_1 \cdot \sqrt{2\pi} + b \cdot \alpha^{-1} + c = 0$ with $a \neq 0$ and $b \neq 0$.*

Proof. Such a relation would give $G_1 \cdot \sqrt{2\pi} = -(b/a)\alpha^{-1} - c/a$, i.e., a $\mathbb{Q}$ -linear relation between $G_1 \cdot \sqrt{2\pi}$ and α^{-1} . Expanding: $1728\sqrt{2}\pi^{-5/2} - \sqrt{2}\pi^{1/2} = -(b/a)(4\pi^3 + \pi^2 + \pi) - c/a$. The left side has nonzero coefficient at $\pi^{-5/2}$ and $\pi^{1/2}$; the right side has zero coefficient at all non-integer powers of π . By the linear independence of distinct π -powers, no such a, b, c exist. $\square$

Assertion 4.1 (Exact numeric verification of impossibility). At `mp.dps = 55`, both sides of (12) are computed explicitly:

$$\text{LHS} = 1728\sqrt{2}\pi^{-5/2} - \sqrt{2}\pi^{1/2} = 137.18939854030114248\dots, \quad (16)$$

$$\text{RHS} = 4\pi^3 + \pi^2 + \pi = 137.03630377587843256\dots, \quad (17)$$

$$\delta = \text{LHS} - \text{RHS} = 0.15309476442270992606\dots \neq 0. \quad (18)$$

The numerical computation and the algebraic argument are mutually consistent.

5 Structural interpretation

5.1 The Coxeter perspective

Assertion 5.1 (G_1 in the $G_2 / j(i)$ frame). The constant $G_1 = (J_{\text{short}}/\pi)^3 - 1$ has a clean Lie-theoretic reading. The integer $J_{\text{short}} = 12$ is simultaneously:

- the Coxeter number $h(G_2) = 12$ of the exceptional Lie algebra G_2 ;
- the number of short roots $|\Phi_{G_2, \text{short}}| = 12$;
- the Coxeter numbers $h(F_4) = h(E_6) = 12$ of the other exceptional algebras sharing this value.

Thus $J_{\text{short}}^3 = 1728 = j(i)$, placing $G_1 + 1$ at the intersection of Lie-theoretic root arithmetic and the modular world. The quantity $G_1 \cdot \sqrt{2\pi}$ introduces the Stirling factor $\sqrt{2\pi}$ into this frame with no Lie-theoretic justification: there is no known representation of $\sqrt{2\pi}$ in terms of root counts or Coxeter numbers of G_2 , F_4 , or E_8 .

Assertion 5.2 (No Hopf-geometric bridge). The TOE $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ arises from the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$ through the three-layer GON architecture (Papers 1–5 of the TOE): the bulk layer contributes $4\pi^3$, the boundary π^2 , and the edge π . The Hopf geometry does not naturally produce factors of $\sqrt{2\pi}$: the volume and surface forms on S^3 involve powers of π with integer exponents. Computing the ratio $G_1 \cdot \sqrt{2\pi} / \text{BREATH} = 0.31867\dots$, where $\text{BREATH} = \pi \cdot \alpha^{-1}$, gives a value close to but not equal to $1/\pi = 0.31831\dots$ (difference $\approx 3.6 \times 10^{-4}$), further confirming the absence of an exact geometric identity.

5.2 The Stirling/Gamma perspective

Assertion 5.3 ($\sqrt{2\pi}$ as Stirling normalisation). The factor $\sqrt{2\pi}$ appears canonically in:

1. Stirling's approximation: $n! \sim \sqrt{2\pi n} (n/e)^n$;
2. the Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$;
3. the functional identity $\sqrt{2\pi} = \sqrt{2} \Gamma(1/2)$ (since $\Gamma(1/2) = \sqrt{\pi}$);
4. the duplication formula: $\Gamma(z) \Gamma(z + 1/2) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)$, giving $\sqrt{2\pi} = 2\Gamma(3/2)\sqrt{2}$.

None of these roles connects $\sqrt{2\pi}$ to the Coxeter number $J_{\text{short}} = 12$ or to the modular value $j(i) = 1728$. The Stirling factor is a normalisation constant for the Gaussian measure; there is no counting argument producing $G_1 \cdot \sqrt{2\pi} = \alpha^{-1}$ from group-theoretic multiplicities.

Assertion 5.4 (The power-of- π mismatch). If one asks what power k would make $G_1 \cdot \pi^k = \alpha^{-1}$ exactly, the answer is

$$k_{\text{exact}} = \frac{\log(\alpha^{-1}/G_1)}{\log \pi} = 0.80178038867288579\dots \quad (19)$$

The Stirling factor $\sqrt{2\pi} = \pi^{k_0}$ corresponds to

$$k_0 = \frac{\log \sqrt{2\pi}}{\log \pi} = \frac{\log \sqrt{2\pi}}{\log \pi} = 0.80275578069914008 \dots \quad (20)$$

The difference $k_0 - k_{\text{exact}} \approx 9.75 \times 10^{-4}$ is the numerical source of the 0.11% gap: $\sqrt{2\pi}$ is slightly the wrong “exponent of π ” to bridge G_1 to α^{-1} . Neither k_{exact} nor k_0 is rational (both are transcendental), so neither yields an exact identity.

5.3 Why the near-miss is convincing without being exact

Assertion 5.5 (Three independent sources of order-137 magnitude). All three expressions independently produce values near 137:

1. $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036$. The dominant term $4\pi^3 \approx 124.025$ plus correction $\pi^2 + \pi \approx 13.011$ gives the total.
2. $G_1 \approx 54.731$. The factor $\sqrt{2\pi} \approx 2.507$ is close to $\alpha^{-1}/G_1 \approx 2.504$, which in turn is close to $\sqrt{2\pi}$. This “near-agreement” is itself a coincidence: α^{-1}/G_1 happens to be close to $\sqrt{2\pi}$ because G_1 is constrained to $[54, 55]$ by the arithmetic of $1728/\pi^3$, and $137/55 \approx 2.491$ is close to $\sqrt{2\pi} \approx 2.507$.
3. The product $G_1 \cdot \sqrt{2\pi}$ lands at 137.189, within 0.11% of α^{-1} .

The coincidence chains: $1728/\pi^3 \approx 55.73$ is an arithmetic near-integer (because $\pi^3 \approx 31$ and $1728 \approx 31 \times 55.7$), and $137/55.7 \approx 2.46$ is close to $\sqrt{2\pi} \approx 2.507$. These two independent near-integer behaviours compound to produce a 0.11% near-miss. No algebraic mechanism is needed.

Assertion 5.6 (Statistical assessment). A uniformly random number in the interval $[135, 140]$ has probability $\approx 2 \times 0.11\% = 0.22\%$ of falling within 0.11% of α^{-1} . The construction $G_1 \cdot \sqrt{2\pi}$ is among the first few dozen natural products of G_1 with special constants one would test. Assuming roughly 50 such constructions tested, the expected number of 0.11% near-misses is $50 \times 0.22\% \approx 0.11$. Thus a 0.11% near-miss among 50 tested constructions is neither surprising nor excluded.

6 Conclusion and open directions

6.1 Verdict

The investigation is complete. The central result is:

Theorem 6.1 (Verdict: COINCIDENCE). *The near-miss $G_1 \cdot \sqrt{2\pi} \approx \alpha^{-1}$ is a numerical coincidence, not an algebraic identity. Specifically:*

- (i) Provably false: *The identity $G_1 \cdot \sqrt{2\pi} = \alpha^{-1}$ is impossible over $\mathbb{Q}$ by the power-structure argument of Theorem 4.1.*
- (ii) PSLQ: *No integer relation for $\delta = G_1 \cdot \sqrt{2\pi} - \alpha^{-1}$ with coefficient norm ≤ 500 exists in any tested basis up to 16 elements, including all natural π -powers, $\sqrt{2\pi}$, $\Gamma(1/4)$, the lemniscate constant, $\log \pi$, $\zeta(3)$, and $e^{2\pi}$.*
- (iii) No correction: *No natural TOE constant T and small integer c satisfy $\delta = c \cdot T$ exactly.*
- (iv) Structural explanation: *The near-miss arises from two compounding arithmetic accidents: $1728/\pi^3 \approx 55.73$ is close to an integer, and $137/55.7 \approx 2.46$ is close to $\sqrt{2\pi} \approx 2.507$. These are independent accidents from genuinely different mathematical domains with no algebraic entanglement.*

Corollary 6.2 (P179 direction resolved). *The open direction recorded as item 3 of Section 6.2 of P179 is resolved: the 0.11% discrepancy between $G_1 \cdot \sqrt{2\pi}$ and α^{-1} has no algebraic structural mechanism, and the near-miss is not evidence of an undiscovered identity.*

6.2 Open directions

1. *Sharper coefficient bound.* Theorem 3.2 certifies no relation with norm ≤ 500 in the tested bases. A Gelfond–Schneider-type analytic argument proving genuine algebraic independence of $G_1 \cdot \sqrt{2\pi}$ and α^{-1} over $\mathbb{Q}$ — beyond the power-structure argument, which proves impossibility of a specific functional form — remains open.
2. *Other G_1 -variants.* P171 introduced $G_1^{(171)} = j(i)/(4\Omega) - 1 \approx 0.003456$, with $\Omega = \pi\alpha^{-1}$ the breath period. The product $G_1^{(171)} \cdot \sqrt{2\pi} \approx 0.00867$ is not close to any known constant. Whether $G_1^{(171)}$ has a Stirling-bridge near-miss of comparable quality to the present one is not investigated here.
3. *Other normalisation factors.* Replacing $\sqrt{2\pi}$ by other Gamma-function normalisations — $\Gamma(1/4)$, $\sqrt{\pi}$, $\Gamma(3/4)$ — in the product with G_1 produces values far from α^{-1} (checked numerically). The $\sqrt{2\pi}$ near-miss appears to be isolated.
4. *The deeper question: why is $1728/\pi^3$ close to an integer?* The near-miss traces ultimately to $1728/\pi^3 \approx 55.73$, i.e., to the fact that $\pi^3 \approx 1728/56 = 30.857$, close to $\pi^3 = 31.006$. This is equivalent to asking why $12^3/(4\Omega_0) \approx 56$, or equivalently $\Omega_0 \approx j(i)/224$. Whether the GON kernel constant $\Omega_0 = \pi^3/4$ has any modular-arithmetic relation to $j(i)/224$ is an open and likely intractable question.

References

- [1] H. R. P. Ferguson, D. H. Bailey and S. Arno, “Analysis of PSLQ, an integer relation finding algorithm,” *Math. Comp.* **68** (1999), 351–369.
- [2] Y. V. Nesterenko, “Modular functions and transcendence questions,” *Mat. Sb.* **187** (1996), 65–96.
- [3] F. Lindemann, “Über die Zahl π ,” *Math. Ann.* **20** (1882), 213–225.
- [4] S. Chowla and A. Selberg, “On Epstein’s zeta function (I),” *Proc. Nat. Acad. Sci. USA* **35** (1949), 371–374.
- [5] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Wiley, 1987.
- [6] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.
- [7] D. H. Bailey and J. M. Borwein, “High-precision arithmetic in mathematical physics,” *Mathematics* **3** (2015), 337–367.