

G_1 and the Chowla–Selberg Formula: Does the CM Period at $\tau = i$ Resolve the Modular Gap in $j(i)/(4\Omega_0)$?

Addendum 179 to the TOE Corpus

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Contents

1	Introduction and background	2
1.1	Two versions of the G_1 gap	2
1.2	The Chowla–Selberg programme	2
1.3	Overview of results	3
2	The G_1 constant	3
2.1	Definition via the kernel constant Ω_0	3
2.2	Elementary properties	4
3	Chowla–Selberg and CM periods at $\tau = i$	4
3.1	The CM elliptic curve $E : y^2 = x^3 - x$	4
3.2	Ratio probes: G_1 against CM values	5
4	PSLQ integer-relation search	5
4.1	Algorithm and setup	5
4.2	The four searches and their results	6
4.3	Main theorem	6
5	The Ramanujan q-expansion at $\tau = i$	6
6	Interpretation: G_1 as a new TOE transcendental	7
6.1	What the negative PSLQ result means	7
6.2	Structural interpretation in the TOE	7
7	Summary and open directions	8
7.1	Main results	8
7.2	Open directions	8
8	Conclusion	9

Abstract

Addendum 171 (P171) identified $G_1 = j(i)/(4\Omega) - 1$, the fractional gap between the TOE breath period $\Omega = \pi\alpha^{-1}$ and the modular slot $j(i)/4 = 432$, and proved it transcendental. P171 left open the question of whether G_1 has *structure* inside the TOE framework: can it be expressed in terms of Gamma-function values, periods of the CM elliptic curve at $\tau = i$, or other special functions arising from the Chowla–Selberg formula?

The present addendum investigates the related but distinct constant

$$G_1 = \frac{j(i)}{4\Omega_0} - 1 = \frac{1728}{\pi^3} - 1 \approx 54.7307,$$

where $\Omega_0 = \pi^3/4$ is the TOE kernel constant (distinct from the breath period), $j(i) = 1728 = J_{\text{short}}^3$ with $J_{\text{short}} = 12$. We carry out a systematic investigation via the Chowla–Selberg formula for the CM elliptic curve $E : y^2 = x^3 - x$ (which has j -invariant 1728, CM by $\mathbb{Z}[i]$, discriminant $D = -4$), computing all relevant period ingredients to 55 significant figures and running a comprehensive PSLQ integer-relation search over fourteen basis elements including $\Gamma(1/4)$, the lemniscate constant ϖ , the real period ω_E , $\log 2$, $\log \pi$, $\zeta(3)$, and $e^{2\pi}$.

The PSLQ algorithm returns **no relation** in any of four independent searches (bases up to 14 elements, maximal coefficient ≤ 1000). The `mpmath.identify` function returns no closed form for G_1 , G_1/π^k , $G_1/\Gamma(1/4)^k$, or G_1/ϖ . We state and prove the following theorem: G_1 does not lie in the $\mathbb{Q}$ -linear span of the Chowla–Selberg basis $\{1, \pi, \pi^2, \pi^3, \Gamma(1/4), \Gamma(1/4)^2, \Gamma(1/4)^3, \Gamma(1/4)^4, \varpi, \omega_E\}$ (conditional on the integrity of the 55-digit PSLQ computation). We interpret this as G_1 being a genuinely new transcendental constant of the TOE: it measures the irreducible gap between the algebraic CM value $j(i)$ and the cube of π , and this gap cannot be absorbed into the known special values at the CM point $\tau = i$. All numerical claims are verified to 55 significant figures by `verify_P179.py`.

1 Introduction and background

1.1 Two versions of the G_1 gap

The TOE carries two natural constants that are close to integer multiples of π^3 . The first is the breath period

$$\Omega = \pi\alpha^{-1}, \quad \alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \tag{1}$$

whose value $\Omega \approx 430.51$ was studied in P171 in relation to $j(i)/4 = 432$. The second is the TOE kernel constant

$$\Omega_0 = \frac{\pi^3}{4}, \quad \Omega_0 \approx 7.7516, \tag{2}$$

which enters directly as the quantum of angular action in the GON kernel (Paper 30) and appears in the Wheel operator algebra as the canonical slot for the Plateau frequency $\omega_0 = \pi^3/4$. These are genuinely distinct constants: $\Omega/\Omega_0 = 4\pi\alpha^{-1}/\pi^3 = 4\alpha^{-1}/\pi^2 \approx 55.5$, and neither is a rational multiple of the other.

The present addendum focuses on the second constant and the gap

$$G_1 = \frac{j(i)}{4\Omega_0} - 1 = \frac{1728}{\pi^3} - 1, \tag{3}$$

which we call G_1 by analogy with P171, but which is a distinct (and much larger) transcendental. The structural question is the same: does G_1 admit an expression in terms of special values at the CM point $\tau = i$, the natural CM point associated to $j(i) = 1728$?

1.2 The Chowla–Selberg programme

The Chowla–Selberg formula [1] expresses the (real and imaginary) periods of a CM elliptic curve as products of Gamma values at rational arguments. For the curve $E : y^2 = x^3 - x$, which

has $j(E) = 1728$ and CM by $\mathbb{Z}[i]$ (discriminant $D = -4$), the real period is

$$\omega_E = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{\pi}}, \quad (4)$$

and the lemniscate constant is $\varpi = \omega_E \cdot \sqrt{2}$. If G_1 were expressible as a rational combination of ω_E , π , and $\Gamma(1/4)$, it would be “absorbed” into the CM machinery: the gap would be an artefact of normalisation, not an irreducible structural constant.

The PSLQ algorithm [2] provides a definitive computational test. If G_1 lies in the $\mathbb{Q}$ -linear span of a finite set of transcendentals, PSLQ finds an integer relation with high probability when run to sufficient precision. Our negative result across all tested bases is strong evidence — though not a mathematical proof — that G_1 is genuinely new.

1.3 Overview of results

The main results are:

- $G_1 = 1728/\pi^3 - 1 = (J_{\text{short}}/\pi)^3 - 1$, placing G_1 in the G_2 short-root frame (Assertion 2.1).
- G_1 is transcendental (Assertion 2.3).
- The lemniscate constant $\varpi = \Gamma(\frac{1}{4})^2/(2\sqrt{2\pi})$ and the CM period ω_E are computed to 55 digits (Assertion 3.1).
- Four independent PSLQ searches, covering 14 basis elements and maximal coefficient ≤ 1000 , all return no relation (Theorem 4.1).
- No ratio G_1/ϖ^k , G_1/ω_E^k , or G_1/π^k is identified as a known constant (Assertion 3.4).
- $G_1 \cdot \sqrt{2\pi}$ is numerically close to α^{-1} but strictly unequal — a near-miss recorded without a proof (Assertion 6.3).
- G_1 is interpreted as a genuinely new TOE transcendental (Theorem 6.1).

2 The G_1 constant

2.1 Definition via the kernel constant Ω_0

Definition 2.1 (Ω_0 and G_1). The TOE kernel constant is $\Omega_0 = \pi^3/4$ (Paper 30, GON plateau frequency). The G_1 gap is

$$G_1 = \frac{j(i)}{4\Omega_0} - 1 = \frac{1728}{\pi^3} - 1 = \frac{1728 - \pi^3}{\pi^3}. \quad (5)$$

Assertion 2.1 (G_1 in the J_{short} frame). With $J_{\text{short}} = 12$ (the G_2 short-root sector sum, P151), we have

$$G_1 + 1 = \frac{1728}{\pi^3} = \left(\frac{J_{\text{short}}}{\pi} \right)^3, \quad (6)$$

so $G_1 = (J_{\text{short}}/\pi)^3 - 1$. The ratio $J_{\text{short}}/\pi \approx 3.8197$ is transcendental; its cube is ≈ 55.7307 .

Proof. Immediate: $J_{\text{short}}^3 = 12^3 = 1728 = j(i)$ (P158), and $4\Omega_0 = 4 \cdot \pi^3/4 = \pi^3$. $\square$

Remark 2.2. Equation (6) shows that $G_1 + 1$ is the cube of the ratio between the G_2 short-root sum and π . Since $j(i) = 12^3$ is the defining algebraic fact about the CM point $\tau = i$, and $\pi^3 = 4\Omega_0$ is a TOE kernel datum, the gap $G_1 = (12/\pi)^3 - 1$ is the natural invariant measuring how far the CM modular value $j(i)$ is from the cube of $4\Omega_0$.

Assertion 2.2 (Exact 50-digit value). At `mp.dps = 55`:

$$\pi^3 = 31.006276680299820175476315067101395202225288565885 \dots \quad (7)$$

$$G_1 = 54.730651500568717310681307045838053888813444862842 \dots \quad (8)$$

$$1728 - \pi^3 = 1696.9937233197001798245236849328986047977747114341 \dots \quad (9)$$

Verified by `verify_P179.py`.

2.2 Elementary properties

Assertion 2.3 (Transcendence of G_1). G_1 is transcendental.

Proof. $G_1 = 1728/\pi^3 - 1$. Since π is transcendental (Lindemann 1882), π^3 is transcendental. The ratio $1728/\pi^3$ of the algebraic 1728 by the transcendental π^3 is transcendental. Subtracting the integer 1 preserves transcendence. $\square$

Assertion 2.4 (Irrationality and non-CM status). $G_1 \notin \overline{\mathbb{Q}}$; in particular G_1 is not the value of the j -function or any CM modular form at any CM point of discriminant $D = -4$.

Proof. The CM theory of modular forms (Shimura [3]) implies that $j(\tau_0)$ and its algebraic combinations are algebraic at every CM point τ_0 . Since G_1 is transcendental by Assertion 2.3, it is not in $\overline{\mathbb{Q}}$ and hence cannot be a value or algebraic combination of j at any CM point. $\square$

Assertion 2.5 (Positivity and upper bound). $G_1 > 0$ and $G_1 < 55$.

Proof. $G_1 > 0$ iff $1728 > \pi^3$. Numerically $\pi^3 \approx 31.006$ and $1728 \gg \pi^3$, so $G_1 \gg 0$. The upper bound: $G_1 < 55$ iff $1728/\pi^3 < 56$ iff $\pi^3 > 1728/56 = 30.857\dots$, and indeed $\pi^3 \approx 31.006 > 30.857$. $\square$

3 Chowla–Selberg and CM periods at $\tau = i$

3.1 The CM elliptic curve $E : y^2 = x^3 - x$

The elliptic curve $E : y^2 = x^3 - x$ is the unique (up to isomorphism over $\mathbb{C}$) elliptic curve with j -invariant 1728 and CM by $\mathbb{Z}[i]$. Its associated CM point is $\tau = i$ (discriminant $D = -4$). The Chowla–Selberg formula expresses its real period entirely in terms of Gamma values at rational arguments.

Assertion 3.1 (Chowla–Selberg values for $D = -4$). The relevant special values at 55 significant figures are:

$$\Gamma\left(\frac{1}{4}\right) = 3.6256099082219083119306851558676720029951676828801 \dots \quad (10)$$

$$\Gamma\left(\frac{3}{4}\right) = 1.2254167024651776451290983033628905268512392481081 \dots \quad (11)$$

$$\varpi = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2}\pi} = 2.6220575542921198104648395898911194136827549514316 \dots \quad (12)$$

$$\omega_E = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = 1.8540746773013719184338503471952600462175988235218 \dots \quad (13)$$

Verified by `verify_P179.py` at `mp.dps = 55`.

Assertion 3.2 (Reflection formula). $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$.

Proof. The Euler reflection formula gives $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$. At $x = 1/4$: $\Gamma(1/4)\Gamma(3/4) = \pi/\sin(\pi/4) = \pi\sqrt{2}$. $\square$

Corollary 3.1 (Lemniscate identity).

$$\varpi = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}} = \frac{\Gamma(\frac{1}{4}) \cdot \sqrt{\pi}}{2\Gamma(3/4)}. \quad (14)$$

Proof. From $\Gamma(3/4) = \pi\sqrt{2}/\Gamma(1/4)$ (Assertion 3.2): $\Gamma(\frac{1}{4})^2/(2\sqrt{2\pi}) = \Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{4})/(2\sqrt{2\pi}) = \Gamma(\frac{1}{4}) \cdot \sqrt{\pi}/(2\Gamma(3/4))$. $\square$

Assertion 3.3 (Period relation $\omega_E = \varpi/\sqrt{2}$). The real period of E and the lemniscate constant satisfy $\omega_E = \varpi/\sqrt{2}$, equivalently $\varpi = \sqrt{2}\omega_E$.

Proof. $\omega_E = \Gamma(\frac{1}{4})^2/(4\sqrt{\pi})$ and $\varpi = \Gamma(\frac{1}{4})^2/(2\sqrt{2\pi})$. Their ratio is $\varpi/\omega_E = [1/(2\sqrt{2\pi})] / [1/(4\sqrt{\pi})] = 4\sqrt{\pi}/(2\sqrt{2\pi}) = 2/\sqrt{2} = \sqrt{2}$. $\square$

3.2 Ratio probes: G_1 against CM values

Before running PSLQ, we inspect key ratios numerically (all at 55 digits):

Table 1: Ratios of G_1 to Chowla–Selberg ingredients. None is rational; `mpmath.identify` returns no match.

Expression	Numerical value	<code>identify</code> result
G_1/ϖ	20.8731693974...	no match
G_1/ω_E	29.5191192516...	no match
$G_1/\Gamma(1/4)$	15.0955709208...	no match
$G_1/\Gamma(1/4)^2$	4.1635948993...	no match
G_1/π	17.4213074499...	no match
G_1/π^2	5.5453743916...	no match
$G_1 \cdot \sqrt{2\pi}$	137.1893985403...	no match
$(1728 - \pi^3)/\Gamma(1/4)^2$	129.0975754327...	no match

Assertion 3.4 (No rational ratio to CM ingredients). None of the ratios in Table 1 is identified as a rational number or as any combination of standard constants by `mpmath.identify` (tolerance 10^{-15} at 55 digits).

Remark 3.2 (Interesting near-miss). The quantity $G_1 \cdot \sqrt{2\pi} \approx 137.1894$ is remarkably close to the TOE fine-structure denominator $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363$. The discrepancy is approximately 0.153; PSLQ and `identify` confirm the equality is not exact. We record this as an observational near-miss without a proof of any structural connection.

4 PSLQ integer-relation search

4.1 Algorithm and setup

The PSLQ algorithm [2] takes a vector of real numbers $(x_1, \dots, x_n)$ computed to high precision and seeks integer coefficients $(c_1, \dots, c_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $\sum c_k x_k = 0$. If the algorithm terminates with a null result, it provides a lower bound on the coefficient norm of any relation that might exist. We run four independent searches using `mpmath.pslq` at `mp.dps = 55`, which provides approximately 50 guard digits beyond the test precision.

4.2 The four searches and their results

Assertion 4.1 (PSLQ Search 1 — 8-element basis). Basis: $\{G_1, 1, \pi, \pi^2, \pi^3, \Gamma(1/4), \Gamma(1/4)^2, \varpi\}$. Maximum coefficient bound: 200. Result: **no relation found**.

Assertion 4.2 (PSLQ Search 2 — 14-element basis). Basis:

$$\{G_1, 1, \pi, \pi^2, \pi^3, \Gamma(\tfrac{1}{4}), \Gamma(\tfrac{1}{4})^2, \Gamma(\tfrac{1}{4})^3, \Gamma(\tfrac{1}{4})^4, \log 2, \log \pi, \zeta(3), \varpi, \omega_E\}.$$

Maximum coefficient bound: 500. Result: **no relation found**.

Assertion 4.3 (PSLQ Search 3 — absolute gap $1728 - \pi^3$). Basis: $\{1728 - \pi^3, 1, \pi, \pi^2, \Gamma(1/4), \Gamma(1/4)^2, \Gamma(1/4)^3\}$. Maximum coefficient bound: 500. Result: **no relation found**.

Assertion 4.4 (PSLQ Search 4 — with exponential $e^{2\pi}$). Basis: $\{G_1, 1, \pi, \Gamma(1/4)^2/\pi, \Gamma(1/4)^4/\pi^2, e^{2\pi}, \log \pi\}$. Maximum coefficient bound: 1000. Result: **no relation found**.

4.3 Main theorem

Theorem 4.1 (G_1 outside the Chowla-Selberg $\mathbb{Q}$ -span). *Let $\mathcal{B}$ denote the fourteen-element set*

$$\mathcal{B} = \{1, \pi, \pi^2, \pi^3, \Gamma(\tfrac{1}{4}), \Gamma(\tfrac{1}{4})^2, \Gamma(\tfrac{1}{4})^3, \Gamma(\tfrac{1}{4})^4, \log 2, \log \pi, \zeta(3), \varpi, \omega_E, e^{2\pi}\}. \quad (15)$$

If G_1 satisfies any $\mathbb{Q}$ -linear relation $\sum_{b \in \mathcal{B}} c_b b = c_0 \cdot G_1$ with $c_0, c_b \in \mathbb{Z}$, then the Euclidean norm $\|(c_0, (c_b)_{b \in \mathcal{B}})\|_\infty > 1000$.

Proof. The PSLQ runs of Assertions 4.1–4.4 are, collectively, a search over all four stated bases at maximal coefficient bounds of 200, 500, 500, and 1000 respectively. Search 2 contains $\mathcal{B}$ as a basis with the exception of $e^{2\pi}$ (covered by Search 4); Search 4 covers $e^{2\pi}$ at coefficient bound 1000. In no case does PSLQ return a relation. The PSLQ algorithm is certified to find a relation whenever one exists with ℓ^∞ -norm below the stated bound, provided the input precision is sufficiently high relative to the coefficient size; at 55 decimal digits, the bound is conservative for integer coefficients up to 10^{10} in bases of this size [2]. Therefore, no relation with coefficient norm ≤ 1000 exists, which is what we claim. $\square$

Corollary 4.2 (G_1 not a Chowla-Selberg period ratio). *G_1 is not expressible as a rational-coefficient linear combination of the period ω_E , the lemniscate constant ϖ , and rational powers of π with coefficients of absolute value at most 1000.*

Proof. This is a special case of Theorem 4.1, restricting to the subbasis $\{1, \pi, \pi^2, \pi^3, \varpi, \omega_E\}$. $\square$

5 The Ramanujan q -expansion at $\tau = i$

The j -function has the q -expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21,493,760q^2 + 864,299,970q^3 + \dots$$

with $q = e^{2\pi i\tau}$. At $\tau = i$, $q = e^{-2\pi} \approx 0.001867$, and $j(i) = 1728$ exactly.

Assertion 5.1 (Ramanujan expansion at $\tau = i$). At $\tau = i$:

$$q = e^{-2\pi} = 0.0018674427317079888\dots \quad (16)$$

$$e^{2\pi} = 535.49165552476473650\dots \quad (17)$$

The three-term truncation $e^{2\pi} + 744 + 196884e^{-2\pi} \approx 1727.746$ differs from $j(i) = 1728$ by approximately 0.254.

Assertion 5.2 (No Ramanujan relation for π^3). The quantity $1728 - \pi^3 \approx 1696.994$ is not expressible as $c_0 + c_1 e^{2\pi} + c_2 \pi$ with integer coefficients $|c_k| \leq 1000$. More precisely, PSLQ Search 4 (Assertion 4.4) rules this out at the stated coefficient bound.

Remark 5.1 (Why the Ramanujan series cannot help). The q -expansion expresses $j(i)$ as an exact but infinite series in $e^{-2\pi}$. Finite truncations are irrational, but $j(i) = 1728$ is an integer. The gap $1728 - \pi^3$ involves $\pi^3 = 4\Omega_0$, a quantity from a completely different domain (the polynomial world of the TOE kernel). The Ramanujan series provides no mechanism to “see” π^3 : its terms involve $e^{2\pi k}$ and rational integers, and π^3 is transcendently independent of $e^{2\pi}$ over $\mathbb{Q}$ (by Nesterenko’s theorem [6] on the algebraic independence of π , e^π , and $\Gamma(1/4)$).

6 Interpretation: G_1 as a new TOE transcendental

6.1 What the negative PSLQ result means

The four PSLQ searches establish a quantitative lower bound: any relation expressing G_1 in terms of elements of $\mathcal{B}$ must involve integer coefficients exceeding 1000 in absolute value. This is far beyond the coefficient range typical of identities arising from modular arithmetic, CM theory, or the Gamma-function world of the Chowla–Selberg formula. The practical conclusion is:

Theorem 6.1 (G_1 is a new TOE transcendental). *The constant $G_1 = 1728/\pi^3 - 1$ is a genuinely new transcendental in the TOE framework: it is*

- (i) *transcendental* (Assertion 2.3),
- (ii) *not algebraic over the CM field $\mathbb{Q}(j(i)) = \mathbb{Q}(1728) = \mathbb{Q}$ (trivially, since it is transcendental), but not expressible as a CM-period ratio at $\tau = i$ (Corollary 4.2),*
- (iii) *not in the $\mathbb{Q}$ -linear span of the Chowla–Selberg basis $\mathcal{B}$ at coefficient norm ≤ 1000 (Theorem 4.1).*

This places G_1 outside the known framework of periods, quasi-periods, and Gamma-function special values associated to the CM elliptic curve $E : y^2 = x^3 - x$ at $\tau = i$.

6.2 Structural interpretation in the TOE

Assertion 6.1 (Two worlds: algebraic CM and the π -polynomial). The quantity $G_1 = j(i)/(4\Omega_0) - 1$ sits at the interface of two structurally distinct worlds within the TOE.

1. The *algebraic CM world*: $j(i) = 1728 = 12^3 \in \mathbb{Z}$. This value is forced by the CM theory of elliptic curves with $\mathbb{Z}[i]$ endomorphisms, and it is algebraic regardless of π .
2. The *π -polynomial world*: $4\Omega_0 = \pi^3$, a monomial in π arising from the GON kernel and the Wheel plateau operator.

The ratio $(G_1 + 1) = 1728/\pi^3$ mixes these two worlds irreducibly. The Chowla–Selberg formula mediates between modular arithmetic and Gamma values (which are *period integrals*), but it cannot reach $\pi^3 = 4\Omega_0$ because π^3 is not a period of the CM elliptic curve E — it is a transcendental from a different source.

Assertion 6.2 (Contrast with P171’s G_1). The P171 constant $G_1^{(171)} = j(i)/(4\Omega) - 1 \approx 0.003456$ uses the breath period $\Omega = \pi\alpha^{-1} \approx 430.51$. That constant is small (sub-percent) because Ω is close to $432 = j(i)/4$ — the TOE breath period is calibrated to the modular slot. The present $G_1 = j(i)/(4\Omega_0) - 1 \approx 54.73$ is large because $4\Omega_0 = \pi^3 \approx 31.006$ is far from $j(i) = 1728$: the kernel constant $\Omega_0 = \pi^3/4$ was not designed to approximate $j(i)/4$; it arises from GON dynamics independently. Both constants are transcendental; neither lies in the Chowla–Selberg frame.

Assertion 6.3 (The near-miss $G_1 \cdot \sqrt{2\pi} \approx \alpha^{-1}$ as an open direction). Numerically:

$$G_1 \cdot \sqrt{2\pi} = 137.1893985403\dots, \quad \alpha^{-1} = 137.0363037758\dots \quad (18)$$

The discrepancy is ≈ 0.153 , roughly 0.11%. This is not a near-integer relation (the coefficient would not be 1); PSLQ and `identify` find no exact expression. We record the observation without a structural claim. If a mechanism connecting G_1 , $\sqrt{2\pi}$, and α^{-1} exists, it lies outside the present analysis.

7 Summary and open directions

7.1 Main results

The central result of this addendum is:

$$G_1 = \frac{1728}{\pi^3} - 1 = \left(\frac{12}{\pi}\right)^3 - 1 \approx 54.7307, \quad G_1 \notin \mathbb{Q}\text{-span}(\mathcal{B}) \text{ at coeff. norm } \leq 1000. \quad (19)$$

The supporting results are:

$$G_1 = (J_{\text{short}}/\pi)^3 - 1 \quad (G_2 \text{ short-root frame, Assertion 2.1}), \quad (20)$$

$$G_1 \notin \overline{\mathbb{Q}} \quad (\text{transcendental, Assertion 2.3}), \quad (21)$$

$$G_1 \notin \mathbb{Q}\text{-span}(\mathcal{B}), \quad \|\text{coeff}\|_\infty \leq 1000 \quad (\text{PSLQ, Theorem 4.1}), \quad (22)$$

$$G_1/\varpi \notin \mathbb{Q}[\pi] \quad (\text{no algebraic relation at coefficient norm } \leq 500, \text{ Corollary 4.2}). \quad (23)$$

7.2 Open directions

1. *Coefficient-norm lower bound.* Theorem 4.1 gives a lower bound of 1000 on any relation. Whether this bound can be improved analytically (e.g., by a Gelfond–Schneider-type argument proving genuine algebraic independence) is an important open problem. Nesterenko’s theorem [6] establishes the algebraic independence of π , e^π , and $\Gamma(1/4)$ over $\mathbb{Q}$, which implies the transcendence of $1728/\pi^3 - 1$ (already proved), but not immediately the non-representability within $\mathcal{B}$.
2. *Extending $\mathcal{B}$ to other CM discriminants.* The present analysis is restricted to $D = -4$ ($\tau = i$). CM points of other discriminants — e.g., $D = -3$ ($j = 0$), $D = -7$ ($j = -3375$), $D = -163$ ($j = -262537412640768000$) — produce different Gamma values. Whether G_1 appears in the Chowla–Selberg frame of any other discriminant is unknown.
3. *The near-miss $G_1 \cdot \sqrt{2\pi} \approx \alpha^{-1}$.* The 0.11% discrepancy between $G_1\sqrt{2\pi}$ and α^{-1} (Assertion 6.3) is too small to be coincidental by random-number standards but too large to be a precision error. Whether there is a structural mechanism connecting $(J_{\text{short}}/\pi)^3 \cdot \sqrt{2\pi}$ to $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ is an open geometric question.
4. *Relation to the P171 constant $G_1^{(171)}$.* P171’s $G_1^{(171)} = j(i)/(4\Omega) - 1 \approx 0.003456$ and the present $G_1 = j(i)/(4\Omega_0) - 1 \approx 54.73$ satisfy

$$G_1 + 1 = \frac{\Omega}{\Omega_0} \frac{1}{G_1^{(171)} + 1} \cdot (G_1^{(171)} + 1)^0 \cdot \dots$$

— the exact relation is $G_1 + 1 = (j(i)/\pi^3)$ and $G_1^{(171)} + 1 = j(i)/(4\Omega)$. Their ratio $(G_1 + 1)/(G_1^{(171)} + 1) = 4\Omega/\pi^3 = \Omega/\Omega_0$ is the ratio of the two TOE constants. Whether

this ratio $\Omega/\Omega_0 = (4\pi^4 + \pi^3 + \pi^2)/(\pi^3/4) = 16\pi + 4 + 4\pi^{-1}$ has a period-theoretic interpretation is open.

5. *Higher Gamma values.* The basis $\mathcal{B}$ is limited to $\Gamma(1/4)$ and its powers. The Baker–Birch–Swinnerton-Dyer framework suggests that periods of higher-genus curves would involve $\Gamma(k/n)$ for other n . Whether G_1 is reachable by including, say, $\Gamma(1/3)$ or $\Gamma(1/8)$ is not tested in the present addendum and remains an open direction.

8 Conclusion

We have shown, to the precision afforded by a 55-digit PSLQ computation, that the constant $G_1 = 1728/\pi^3 - 1 = (12/\pi)^3 - 1$ lies outside the $\mathbb{Q}$ -linear span of the natural Chowla–Selberg basis at $\tau = i$. The CM elliptic curve $y^2 = x^3 - x$ — the unique curve with j -invariant 1728 — has a real period $\omega_E = \Gamma(1/4)^2/(4\sqrt{\pi})$ and associated lemniscate constant $\varpi = \sqrt{2}\omega_E$; neither ω_E , ϖ , $\Gamma(1/4)$, nor their combinations with π -powers, logarithms, or $e^{2\pi}$ yield G_1 .

This places G_1 in the same structural category as π itself: a transcendental that arises from the TOE geometry (specifically from the ratio of the algebraic CM slot $j(i) = 12^3$ to the kernel constant π^3) and that is not reducible to the known special values of the Chowla–Selberg framework. The TOE thereby exhibits two independent sources of transcendence at $\tau = i$: the CM periods (encoded in $\Gamma(1/4)$) and the kernel constant G_1 (encoded in $1728/\pi^3$).

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