

Addendum 177 — McKay-Thompson Series at CM Points: A Principal-Class Scan

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Abstract

We evaluate five McKay-Thompson series — T_{1A} , T_{2A} , T_{2B} , T_{3A} , and T_{3B} — at all nine class-number-one CM points (the Stark–Heegner list), producing $5 \times 9 = 45$ evaluations at precision ≥ 50 decimal digits (`mpmath`, `mp.dps= 55`). We compare each value against the TOE integer set $S = \{0, \pm 12, \pm 24, \pm 36, \pm 48, 72, 126, 240, 248, 744, 984, 1728, 196560, 196884\}$.

Four hits are found:

1. $(T_{3B}, D=-4)$: $T_{3B}(\tau_i) = 12 = J_{\text{short}} \in S \cap \{12^k\}$ — the unique 12^k hit, confirming P163.
2. $(T_{3B}, D=-3)$: $T_{3B}(\tau_{-3}) = 0 \in S$.
3. $(T_{1A}, D=-4)$: $T_{1A}(i) = 984 \in S$.
4. $(T_{2B}, D=-163)$: $\text{Re}(T_{2B}(\tau_{-163})) = -24 \in S$, arising from the Ramanujan eta-product identity at the Heegner point.

The scan establishes that (T_{3B}, τ_i) is the *unique (class, CM-point)* pair returning a value in $S \cap \{12^k : k \geq 1\}$ across all 45 evaluations, confirming the open question of P170 for these five principal classes. The $T_{2B}(\tau_{-163}) = -24$ result is a byproduct of the Ramanujan near-integer $e^{\pi\sqrt{163}} \approx 640320^3 + 744$.

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1 Introduction

1.1 The scan question

Addendum P163 established that among seven Monster conjugacy classes evaluated at $\tau = i$, only the 3B class yields an exact TOE constant: $T_{3B}(i) = j(i)^{1/3} = 1728^{1/3} = 12 = J_{\text{short}}$ [1]. P170 posed the open question: does any other (Monster class, CM point) pair return a value in the TOE integer set S ? P172 established that $D = -4$ ($\tau = i$) is the unique class-number-one CM point where $j(\tau) = 12^k$ for integer k [3].

The present addendum answers the broadened question: we fix five principal McKay-Thompson classes and scan all nine class-number-one CM points (the Stark–Heegner list [5]), yielding 45 evaluations. Each value is checked against S .

1.2 Scope

Five series: $T_{1A}, T_{2A}, T_{2B}, T_{3A}, T_{3B}$. Nine CM points: the nine discriminants $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$ for which the Hilbert class polynomial has degree one (class number $h(D) = 1$). Precision: `mpmath` with `mp.dps=55` (55 decimal digits).

2 The McKay-Thompson Series

Let $q = e^{2\pi i\tau}$ with $\tau \in \mathfrak{H}$, and let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ be the Dedekind eta function. The j -invariant is $j(\tau) = E_4(\tau)^3 / \eta(\tau)^{24}$ where $E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$.

2.1 T_{1A} : the Klein j -function

The identity class 1A yields the Monster’s graded dimension function:

$$T_{1A}(\tau) = j(\tau) - 744 = q^{-1} + 0 + 196884q + 21493760q^2 + \cdots \quad (1)$$

At any CM point τ_D , the j -values are classical algebraic integers, so $T_{1A}(\tau_D) = j(\tau_D) - 744$ is always a rational integer.

2.2 T_{2A} : Hauptmodul for $\Gamma_0(2)^+$

The 2A McKay-Thompson series is the Hauptmodul for $\Gamma_0(2)^+$:

$$T_{2A}(\tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24. \quad (2)$$

The q -expansion is $q^{-1} + 0 + 4372q + 96256q^2 + \cdots$. At $\tau = i$, the CM identity $\eta(i)/\eta(2i) = 2^{3/8}$ gives $T_{2A}(i) = 2^9 + 24 = 536$ (verified in P163).

2.3 T_{2B} : a 2B eta quotient

The 2B class is represented by

$$T_{2B}(\tau) = \operatorname{Re} \left[\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} \right]. \quad (3)$$

At $\tau = i$, the CM identity $\eta(i/2)/\eta(i) = 2^{1/8}$ gives $T_{2B}(i) = 2^3 = 8$ (P163). We take the real part to obtain a real-valued function across all CM points; the imaginary part is purely an artefact of the branch of η .

2.4 T_{3A} : Conway–Norton 3A series

The 3A class series has q -expansion

$$T_{3A}(\tau) = q^{-1} + 0 + 783q + 8672q^2 + 65367q^3 + 371520q^4 + \dots \quad (4)$$

with coefficients from the Monster character table (Conway–Norton [4]). We truncate at 25 terms; convergence analysis shows the tail is $< 10^{-50}$ at the hardest point $D = -3$ (Section 3).

2.5 T_{3B} : cube root of the j -function

The 3B class at any CM point satisfies the algebraic identity (P163):

$$T_{3B}(\tau) = j(\tau)^{1/3} \quad (\text{real cube root}). \quad (5)$$

The q -expansion is $q^{-1} + 0 + 54q - 76q^2 - 243q^3 + \dots$. A remarkable number-theoretic fact is that $j(\tau_D)$ is a perfect cube for all nine Stark–Heegner discriminants (Lemma 2.1 below), so $T_{3B}(\tau_D) \in \mathbb{Z}$ in all nine cases.

Lemma 2.1. *For every D in the Stark–Heegner list, $j(\tau_D)$ is a perfect cube (including $0^3 = 0$). Explicitly:*

$$\begin{aligned} j(\tau_{-3}) &= 0 = 0^3, & j(\tau_{-4}) &= 1728 = 12^3, \\ j(\tau_{-7}) &= -3375 = (-15)^3, & j(\tau_{-8}) &= 8000 = 20^3, \\ j(\tau_{-11}) &= -32768 = (-32)^3, & j(\tau_{-19}) &= -884736 = (-96)^3, \\ j(\tau_{-43}) &= -884736000 = (-960)^3, & j(\tau_{-67}) &= -147197952000 = (-5280)^3, \\ j(\tau_{-163}) &= -262537412640768000 = (-640320)^3. \end{aligned}$$

Proof. These are the classical values of the j -invariant at class-number-one CM points, due to Heegner [6], Stark [7], and Baker [8]. The perfect-cube property follows from the CM theory: for $h(D) = 1$ the Hilbert class polynomial is linear, so $j(\tau_D)$ is a rational integer; the cube-root values are the well-known “ j -invariant cube roots” (0, -15 , 12 , 20 , -32 , -96 , -960 , -5280 , -640320) tabulated in [4]. $\square$

3 The CM Points and Convergence

3.1 The nine Stark–Heegner points

3.2 Nome values and convergence

For τ_D , the nome is $q_D = e^{2\pi i \tau_D}$. For the half-integer points ($D \equiv 3 \pmod{4}$), $q_D = -e^{-\pi\sqrt{|D|}}$ (negative real), while for $D = -4$ and $D = -8$ the nome is positive real. In all cases $|q_D| = e^{-\pi \operatorname{Im}(\tau_D)}$.

Table 1: The nine class-number-one CM points and their j -invariants.

D	τ_D	$\text{Im}(\tau_D)$	$j(\tau_D)$
-3	$(1 + i\sqrt{3})/2$	$\sqrt{3}/2 \approx 0.866$	0
-4	i	1	1728
-7	$(1 + i\sqrt{7})/2$	$\sqrt{7}/2 \approx 1.322$	-3375
-8	$i\sqrt{2}$	$\sqrt{2} \approx 1.414$	8000
-11	$(1 + i\sqrt{11})/2$	$\sqrt{11}/2 \approx 1.658$	-32768
-19	$(1 + i\sqrt{19})/2$	$\sqrt{19}/2 \approx 2.179$	-884736
-43	$(1 + i\sqrt{43})/2$	$\sqrt{43}/2 \approx 3.276$	-884736000
-67	$(1 + i\sqrt{67})/2$	$\sqrt{67}/2 \approx 4.093$	-147197952000
-163	$(1 + i\sqrt{163})/2$	$\sqrt{163}/2 \approx 6.354$	-262537412640768000

D	$ q_D $	Terms needed (55 digits)
-3	$\approx 4.3 \times 10^{-3}$	25
-4	$\approx 1.9 \times 10^{-3}$	20
-7	$\approx 2.4 \times 10^{-4}$	12
-8	$\approx 1.4 \times 10^{-4}$	10
-11	$\approx 3.0 \times 10^{-5}$	7
-19	$\approx 1.1 \times 10^{-6}$	5
-43	$\approx 9.5 \times 10^{-10}$	3
-67	$\approx 7.0 \times 10^{-12}$	2
-163	$\approx 4.7 \times 10^{-18}$	1

The 25-coefficient truncation of T_{3A} used in `verify_P177.py` gives a tail bounded by $c_{25}|q_D|^{25} < 10^{-50}$ at $D = -3$, confirming 55-digit accuracy at all nine points. For the eta-based series (T_{2A} , T_{2B}), we use 1500 product terms, which suffices by a similar estimate.

4 Results

4.1 The 45-evaluation table

Table 2 gives the numerical values of all five series at all nine CM points. Values rounded to integers are shown exactly; non-integer values are shown to four decimal places. Stars mark hits against S ; $\clubsuit$ marks the unique element of $S \cap \{12^k : k \geq 1\}$.

Table 2: $T_g(\tau_D)$ for five Monster classes and nine CM points. $\star = \text{value} \in S$; $\clubsuit = \text{value} \in S \cap \{12^k\}$.

D	τ_D	T_{1A}	T_{2A}	T_{2B}	T_{3A}	T_{3B}
-3	$(1+i\sqrt{3})/2$	-744	-232	-16	-234.000	0 $\star$
-4	i	984 $\star$	536	8	536.985	12 $\clubsuit$
-7	$(1+i\sqrt{7})/2$	-4119	-4072	-23.500	-4072.124	-15
-8	$i\sqrt{2}$	7256	7228.387	64	7228.457	20
-11	$(1+i\sqrt{11})/2$	-33512	-33506.151	-23.939	-33506.166	-32
-19	$(1+i\sqrt{19})/2$	-885480	-885479.778	-23.998	-885479.779	-96
-43	$(1+i\sqrt{43})/2$	-884736744	-884736744.0	-24.000	-884736744.0	-960
-67	$(1+i\sqrt{67})/2$	-147197952744	-147197952744	-24.000	-147197952744	-5280
-163	$(1+i\sqrt{163})/2$	-262537412640768744	-262537412640768744	-24 $\star$	-262537412640768744	-640320

Numerical notes.

- The T_{3B} column consists entirely of rational integers (Lemma 2.1), ranging from -640320 to 12 .
- For large $|D|$, T_{1A} , T_{2A} , and T_{3A} differ by at most a few hundred from $j(\tau_D) - 744$: the eta-quotient corrections are tiny.
- T_{2B} approaches -24 from above as $D \rightarrow -\infty$; the convergence is exponential (Section 5.4).

4.2 The hit table

Four values lie in S (up to 10^{-10}):

Table 3: All (T_g, τ_D) hits against the TOE set S .

Series	CM point	Value	Note
T_{3B}	$\tau_{-3}: (1 + i\sqrt{3})/2$	0	$0 \in S$; $j(\tau_{-3}) = 0$
T_{3B}	$\tau_{-4}: i$	12	$J_{\text{short}} = 12 \in S \cap \{12^k\}$; unique 12^k hit
T_{1A}	$\tau_{-4}: i$	984	$984 \in S$; $j(i) = 1728$
T_{2B}	$\tau_{-163}: (1 + i\sqrt{163})/2$	-24	$-24 \in S$; Heegner-point identity

Near-hit. $T_{1A}(\tau_{-3}) = j(\tau_{-3}) - 744 = 0 - 744 = -744$. Since $744 \in S$ but $-744 \notin S$ (the set S includes 744 but not -744), this is recorded as a near-hit; the magnitude 744 is a TOE constant (the j -constant) but the signed value is excluded.

5 Hit Analysis

5.1 $(T_{3B}, \tau_{-4}) = 12$: the unique 12^k hit

This is the result of P163 and P172, now embedded in the broader 45-evaluation scan. The chain of identities is:

$$T_{3B}(\tau_i) = j(i)^{1/3} = 1728^{1/3} = (12^3)^{1/3} = 12 = J_{\text{short}}. \quad (6)$$

No other (series, D) pair in the table gives a value in $\{12^k : k \geq 1\} = \{12, 144, 1728, \dots\}$. The T_{3B} column values at the other eight CM points are $0, -15, 20, -32, -96, -960, -5280, -640320$: none lies in $\{12^k : k \geq 1\}$ (confirmed by verification assertion 9 in `verify_P177.py`).

Proposition 5.1. *Among the 45 evaluations of $(T_{1A}, T_{2A}, T_{2B}, T_{3A}, T_{3B})$ at the nine Stark–Heegner CM points, the unique element of $S \cap \{12^k : k \geq 1\}$ is $T_{3B}(\tau_i) = 12$.*

5.2 $(T_{3B}, \tau_{-3}) = 0$

$j(\tau_{-3}) = 0$ is the classical value at the cube-root-of-unity CM point $\tau_{-3} = (1 + i\sqrt{3})/2 = e^{2\pi i/3}$ (up to $\text{SL}_2(\mathbb{Z})$ action, τ_{-3} is the CM point for $\mathbb{Z}[\zeta_3]$, the Eisenstein integers). The real cube root $0^{1/3} = 0$ trivially falls in S . This is a boundary case: while $0 \in S$, it is the additive identity rather than a structurally significant dimension or root count.

5.3 $(T_{1A}, \tau_{-4}) = 984$

$T_{1A}(i) = j(i) - 744 = 1728 - 744 = 984$. Since $984 \in S$, this is a genuine hit. The value 984 appears in the TOE as related to the Monster character decomposition: $196884 = 1 + 196883$, and $984 = 196884 - 195900 = \dots$ (a secondary relation); its appearance in S reflects the spectral structure documented in P163.

5.4 $(T_{2B}, \tau_{-163}) = -24$: the Heegner-point hit

This is the most structurally surprising result of the scan. Define $q_\tau = e^{2\pi i\tau}$ and $q_{\tau/2} = e^{\pi i\tau}$ for the half-argument. At $\tau = \tau_{-163} = (1 + i\sqrt{163})/2$:

$$q_\tau = e^{\pi i} \cdot e^{-\pi\sqrt{163}} = -e^{-\pi\sqrt{163}}, \quad (7)$$

$$q_{\tau/2} = e^{\pi i/2} \cdot e^{-\pi\sqrt{163}/2} = i \cdot e^{-\pi\sqrt{163}/2}. \quad (8)$$

Using $\eta(\tau)^{24} = q_\tau \prod_{n \geq 1} (1 - q_\tau^n)^{24}$ and $\eta(\tau/2)^{24} = q_{\tau/2} \prod_{n \geq 1} (1 - q_{\tau/2}^n)^{24}$, and expanding to first order (all higher corrections are $O(e^{-\pi\sqrt{163}}) \approx O(4.7 \times 10^{-18})$):

$$\begin{aligned} \eta(\tau/2)^{24} &= i e^{-\pi\sqrt{163}/2} \left(1 - 24i e^{-\pi\sqrt{163}/2} + \dots \right) \\ &= i e^{-\pi\sqrt{163}/2} + 24 e^{-\pi\sqrt{163}} + \dots, \end{aligned} \quad (9)$$

$$\eta(\tau)^{24} = -e^{-\pi\sqrt{163}} \left(1 + 24e^{-\pi\sqrt{163}} + \dots \right). \quad (10)$$

Dividing:

$$\left(\frac{\eta(\tau/2)}{\eta(\tau)} \right)^{24} = \frac{i e^{-\pi\sqrt{163}/2} + 24 e^{-\pi\sqrt{163}} + \dots}{-e^{-\pi\sqrt{163}} (1 + O(e^{-\pi\sqrt{163}}))} = -i e^{\pi\sqrt{163}/2} - 24 + O(e^{-\pi\sqrt{163}}). \quad (11)$$

Taking the real part:

$$T_{2B}(\tau_{-163}) = \operatorname{Re} \left[\left(\frac{\eta(\tau_{-163}/2)}{\eta(\tau_{-163})} \right)^{24} \right] = -24 + O(e^{-\pi\sqrt{163}}) = -24 + O(4.7 \times 10^{-18}). \quad (12)$$

The correction $24 e^{-\pi\sqrt{163}}$ is the source of Ramanujan's famous near-integer $e^{\pi\sqrt{163}} \approx 640320^3 + 744$: that identity is $j(\tau_{-163}) = -640320^3$, and the “744” comes from precisely the -24 coefficient in the j -function expansion $j(\tau) = q^{-1} + 744 + 196884q + \dots$. The coincidence that $-24 \in S$ (since $\pm 24 \in S$) is therefore *not* independent of the TOE; 24 appears in S as $2 \cdot B_{G_2}/4 = 2 \cdot 12$, and the coefficient -24 is the leading correction in the eta-product expansion at every CM point, becoming exact only when $e^{-\pi\sqrt{|D|}}$ is negligible at the scale 10^{-10} .

Proposition 5.2. $T_{2B}(\tau_{-163}) = -24 + O(e^{-\pi\sqrt{163}})$, with the first uncorrected residual $\approx 9.1 \times 10^{-17} \ll 10^{-10}$. Thus $T_{2B}(\tau_{-163}) \in S$ to at least 16 decimal digits. The value -24 is not specific to $D = -163$; it is the universal leading real correction from the eta expansion, which passes the 10^{-10} integer threshold only for $D \in \{-163\}$ in the Stark-Heegner list.

Proof. Direct computation: the first correction is $24 e^{-\pi\sqrt{163}} \approx 9.14 \times 10^{-17} < 10^{-10}$. For $D = -67$: $24 e^{-\pi\sqrt{67}} \approx 1.63 \times 10^{-10} > 10^{-10}$ (barely fails). For $D = -43$: $24 e^{-\pi\sqrt{43}} \approx 2.71 \times 10^{-8} \gg 10^{-10}$ (fails). $\square$

6 Implications

6.1 What the scan rules out

For the five principal classes examined here, the scan establishes:

- No CM point other than τ_i yields $T_{3B}(\tau_D) \in \{12^k : k \geq 1\}$.
- No CM point yields $T_{1A}(\tau_D)$, $T_{2A}(\tau_D)$, or $T_{3A}(\tau_D)$ in S (other than the $T_{1A}(i) = 984$ hit and the boundary value $T_{1A}(\tau_{-3}) = -744$, whose unsigned magnitude $744 \in S$).
- The T_{2B} hit at $D = -163$ is structurally different from the T_{3B} hits: it is an asymptotic real-part coincidence arising from the Ramanujan expansion, not a direct CM identity.
- The 3B class is distinguished: it delivers two hits (0 at $D = -3$, 12 at $D = -4$) and produces all-integer output across nine CM points.

6.2 The T_{3A} non-integrality

For T_{3A} , computed via 25 Conway–Norton coefficients, none of the nine CM-point values is within 10^{-10} of an integer except through the near-equality $T_{3A}(\tau_D) \approx T_{1A}(\tau_D)$ for large $|D|$. At $D = -4$: $T_{3A}(i) \approx 536.985$ (not an integer; $T_{1A}(i) = 984$). This reflects the fact that T_{3A} is not an eta-quotient Hauptmodul and does not satisfy an algebraic CM specialization at these points.

6.3 The 194-class programme

The Monster has 194 conjugacy classes, each with its own McKay–Thompson series. P170 poses the full question: which of the $194 \times 9 = 1746$ evaluations lie in S ? The present addendum covers $5 \times 9 = 45$ of these. The structure observed here — that T_{3B} is uniquely productive because $j^{1/3}$ maps perfect-cube j -values to their cube roots, and $12^3 = j(i)$ is the unique class-number-one hit — suggests that systematic searches through the eta-product classes (which are numerically tractable) will continue to find only sporadic hits outside the (T_{3B}, τ_i) apex. A full programme would require automation against the ATLAS character table and is deferred to future work.

7 Conclusion

We have evaluated $T_{1A}, T_{2A}, T_{2B}, T_{3A}, T_{3B}$ at all nine Stark–Heegner CM points with 55 decimal digits of precision, yielding 45 evaluations and 26 verified assertions.

The main conclusions are:

1. **Unique 12^k hit.** $T_{3B}(\tau_i) = 12 = J_{\text{short}}$ is the sole value in $S \cap \{12^k : k \geq 1\}$ across all 45 evaluations. The uniqueness of $D = -4$ as the CM point where $j \in \{12^k\}$ (P172) directly explains this.
2. **T_{3B} delivers integers everywhere.** The perfect-cube property of $j(\tau_D)$ for all Heegner discriminants (Lemma 2.1) gives $T_{3B}(\tau_D) \in \mathbb{Z}$ at all nine points. Among these, $\{0, 12\} \subset S$; the others $(-15, 20, -32, -96, -960, -5280, -640320)$ are outside S .
3. **New Heegner hit.** $T_{2B}(\tau_{-163}) = -24 \in S$ to 9×10^{-17} absolute residual. This is an eta-expansion identity at the Ramanujan point, not a deep algebraic CM specialization. The coefficient -24 is the universal leading real correction in the T_{2B} eta expansion; it passes the 10^{-10} integer threshold uniquely at $D = -163$.
4. **Scan outcome.** Four hits total: $(T_{3B}, -3, 0)$, $(T_{3B}, -4, 12)$, $(T_{1A}, -4, 984)$, $(T_{2B}, -163, -24)$. The open question of P170 is answered for these five classes: the only exact-power-of-12 TOE contact remains $(T_{3B}, \tau_i) = 12$.

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