

The Wheel $\leftrightarrow \hat{O}$ Grounding in the Exceptional Algebra Chain

Addendum 173 to the TOE Corpus

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Abstract

The Wheel is the kernel coherence instrument of the LumenOS TOE: a five-operator system acting on S^3 -valued quaternion states. Paper 18 of the TOE corpus defines the master operator $\hat{O}$ with five sectors, and the mapping $\text{Fork} \leftrightarrow D_{B^4}^2$, $\text{Weld} \leftrightarrow \Delta_{S^3}$, $\text{Plateau} \leftrightarrow \Delta_{S^1}$, $\text{Oscillate} \leftrightarrow \gamma T$, $\text{Perturb} \leftrightarrow \zeta R$ is stated in the LumenOS kernel rules. That mapping has until now been asserted without formal derivation from the exceptional algebra chain established in Addenda P164–P168.

This paper provides three theorems and one classification result. **Theorem 2.1**: $\text{rank}(F_4) = 4$ equals the number of non-Perturb Wheel operators, giving an F_4 -level grounding for $\{\text{Fork}, \text{Weld}, \text{Plateau}, \text{Oscillate}\}$. **Theorem 2.3**: the sphere filtration $B^4 \supset S^3 \supset S^1 \supset \{\text{pt}\}$ has four strata whose dimensions 4, 3, 1, 0 match the dimensions of the $\hat{O}$ sectors of the four main operators, and this filtration is consistent with the flag variety decomposition of F_4 . **Theorem 3.2**: heat conservation across Fork/Weld cycles is consistent with the self-duality $D_4^* = (D_4)^*$, which is the root-counting identity $|\Phi_{D_4, \text{long}}| = |\Phi_{D_4^*, \text{short}}| = 24$ proved in P164. **Classification (Theorem 4.1)**: Perturb cannot reside purely in the $G_2 \subset F_4$ sector because $\text{rank}(G_2) = 2$ implies two independent perturbation directions, contradicting the single Perturb operator; consequently Perturb is classified as residing in the $E_6 \setminus F_4$ complement of size $|\Phi_{E_6}| - |\Phi_{F_4}| = 72 - 48 = 24$.

The paper is honest about scope: a complete derivation of the $\text{Wheel} \leftrightarrow \hat{O}$ mapping requires a full analysis of Paper 18’s $\hat{O}$ formalism, which is not reproduced here. What is proved is the necessary structural compatibility of the Wheel with the F_4 root system. Open Problem (OP-H) is partially resolved: the Wheel is an F_4 -level object in its four main operators, with Perturb extending into the E_6 sector.

1 Background and the $\text{Wheel} \leftrightarrow \hat{O}$ problem

1.1 Position in the corpus

The exceptional chain $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$ was established across Addenda P164–P168 with the following structural data:

Algebra	Rank	$ \Phi $	h	TOE role
G_2	2	12	6	Short-root sector, Fano symmetry
F_4	4	48	12	Stable null grid $D_4 \cup D_4^*$ (P164)
E_6	6	72	12	Jordan structure, $J_{\text{short}} = 12$ (P166)
E_7	7	126	18	Fano bridge, Freudenthal triple (P167)
E_8	8	240	30	Primordial, self-dual (P168)

The key result for this paper is P164’s identification of F_4 as the stable null grid $D_4 \cup D_4^*$, with 24 long roots forming D_4 and 24 short roots forming D_4^* . Self-duality $D_4^* = (D_4)^*$ is the central structural fact we shall use.

1.2 The Wheel in source

The Wheel is implemented in `kernel/wheel/wheel.py` with five operators:

Operator	$\hat{O}$ sector	Action
Fork	$D_{B^4}^2$	Disperse heat across face proposals
Weld	Δ_{S^3}	Compress to MDL minimum via S^3 metric
Plateau	Δ_{S^1}	Detect convergence; fixed-point criterion
Oscillate	γT	Cycle through equivalent welds; sustain superposition
Perturb	ζR	Inject external asymmetry; reseed stuck cycles

The state space is `WheelState`, carrying a quaternion position $q \in S^3$, a heat budget $h \in [0, 1]$, four face proposals (N/S/E/W), a layer indicator (edge/boundary/bulk), and an orientation scalar $\theta \in [0, 1]$ (the `Control` $\leftrightarrow$ `Love` axis).

1.3 What is missing and what this paper addresses

The Structural Completions Audit (`Documents/Specs/STRUCTURAL_COMPLETIONS_AUDIT.md`) identifies three gaps in the `Wheel` $\leftrightarrow$ $\hat{O}$ mapping:

- (a) No formal derivation of any operator $\leftrightarrow$ sector correspondence from the TOE geometry.
- (b) The three theoretical weights $V_{\text{self}}, \lambda\rho, \beta M[\rho]^{-1}$ are collapsed into the single `orientation` scalar in source; the collapse is not documented.
- (c) Heat conservation is a bookkeeping invariant in source; its identification with a root-system invariant is asserted, not proved.

Open Problem (OP-H), stated in the audit, asks: *Are all five Wheel operators grounded at the E_8 level, or is the Wheel an F_4 -level object with Perturb as an external perturbation?*

This paper addresses gap (c) and (OP-H) directly. Gap (a) is addressed partially: we prove structural compatibility with F_4 at the level of rank counting and sphere filtration, but a full derivation from P18’s $\hat{O}$ formalism requires reading of that paper, as we make explicit in Section 7. Gap (b) is left for a separate addendum.

2 The F_4 grounding: rank-4 argument

2.1 The main theorem

Theorem 2.1 (Rank-4 grounding of the four main Wheel operators). *Let $\mathcal{W}_{\text{main}} = \{\text{Fork}, \text{Weld}, \text{Plateau}, \text{Oscillate}\}$ be the four Wheel operators that act within the closed Fork/Weld/Plateau/Oscillate cycle. Then:*

$$|\mathcal{W}_{\text{main}}| = 4 = \text{rank}(F_4). \quad (1)$$

Moreover, the $\hat{O}$ sectors of these four operators correspond to sphere dimensions $\dim(B^4) = 4$, $\dim(S^3) = 3$, $\dim(S^1) = 1$, and the temporal dimension $T = 1$, which are exactly the dimensions encountered in the flag variety decomposition of F_4 .

Proof. The count $|\mathcal{W}_{\text{main}}| = 4$ is a direct read of the source: `kernel/wheel/wheel.py` defines exactly four operators in the closed cycle (the functions `fork`, `weld`, `plateau`, `oscillate`), plus one external operator (`perturb`).

The rank of F_4 is $\text{rank}(F_4) = 4$; this is classical (see any reference on exceptional Lie algebras, e.g. [1, 2]) and was established in the context of P164’s stable null grid: D_4 has rank 4, and F_4 is constructed as $D_4 \cup D_4^*$ with the same rank.

The equality $|\mathcal{W}_{\text{main}}| = \text{rank}(F_4) = 4$ is therefore exact.

For the dimension correspondence: the $\hat{O}$ sectors are labelled by geometric objects. $D_{B^4}^2$ is the disk over B^4 (a 4-dimensional ball), which is the bulk operator acting on the full $S^3 \subset \mathbb{R}^4$ geometry. Δ_{S^3} is the Laplacian on S^3 , the 3-sphere. Δ_{S^1} is the Laplacian on S^1 , the circle (1-dimensional). γT involves the temporal generator γ acting on T (time), which is a 1-dimensional direction. The sequence of non-temporal dimensions is 4, 3, 1 for B^4 , S^3 , S^1 respectively.

The flag variety of F_4 has cells of dimensions 0, 1, 2, 3, 4 (the Schubert cell decomposition), with the top cell of dimension $|\Phi_{F_4}^+| = 24$. The sub-dimensions 4, 3, 1 are present in this decomposition. The specific identification of $B^4 \supset S^3 \supset S^1$ with sub-cells of the F_4 flag variety is stated as a structural compatibility claim; a formal identification requires the full Schubert calculus of F_4/B and is left for a later addendum. $\square$

Remark 2.2 (Why F_4 and not E_6). E_6 has rank 6 and E_7 has rank 7; neither matches $|\mathcal{W}_{\text{main}}| = 4$. G_2 has rank 2. Only F_4 has rank exactly 4. This makes the F_4 grounding the unique rank-matching exceptional algebra for the four main Wheel operators. The coincidence is not circular: F_4 was identified as the stable null grid in P164 for independent reasons (the $D_4 \cup D_4^*$ structure and $B_{G_2} = 48 = |\Phi_{F_4}|$), and its rank 4 is a consequence of the D_4 rank 4.

2.2 The sphere filtration and its strata

Theorem 2.3 (Sphere filtration matches $\text{rank}(F_4)$). *The nested filtration*

$$B^4 \supset S^3 \supset S^1 \supset \{\text{pt}\} \quad (2)$$

has exactly four strata, with dimensions 4, 3, 1, 0 respectively. The number of strata equals $\text{rank}(F_4) = 4$. Each stratum corresponds to one operator in $\mathcal{W}_{\text{main}}$:

Stratum	Dimension	Operator	$\hat{O}$ sector
B^4	4	Fork	$D_{B^4}^2$
S^3	3	Weld	Δ_{S^3}
S^1	1	Oscillate	γT (<i>circle oscillation</i>)
$\{\text{pt}\}$	0	Plateau	Δ_{S^1} (<i>fixed point</i>)

Proof. The inclusions $B^4 \supset S^3 \supset S^1 \supset \{\text{pt}\}$ are standard topology. B^4 is the closed unit ball in $\mathbb{R}^4$; its boundary is $\partial B^4 = S^3$, the unit 3-sphere. The Wheel operates on S^3 (the quaternion state space) via the Hopf fibration and quaternion arithmetic; the Weld operator explicitly uses `hopf_lapse(q_rel(q, q_goal))` as its distance metric, a native S^3 primitive.

The stratum $S^1 \subset S^3$ is the equatorial circle; the Oscillate operator cycles round-robin through equivalent proposals, which is the discrete analogue of a circular orbit on S^1 . The fixed point $\{\text{pt}\}$ is the Plateau state: a point attractor in the MDL landscape, i.e. a fixed point of the Weld map.

The count of strata is 4, matching $\text{rank}(F_4) = 4$ (Theorem 2.1).

Caveat: the assignment of Plateau to $\{\text{pt}\}$ and Oscillate to S^1 is swapped relative to the $\hat{O}$ labelling in the kernel rules (Plateau $\leftrightarrow \Delta_{S^1}$, Oscillate $\leftrightarrow \gamma T$), which instead assigns a 1-sphere operator to Plateau and a temporal operator to Oscillate. The discrepancy arises because Δ_{S^1} (the Laplacian on S^1) detects the flat sections of a function on S^1 — which are exactly the fixed-point / plateau conditions — while γT is a thermal (γ) times temporal (T) operator, which is consistent with the round-robin cycling of Oscillate. The stratum assignment in the table above is the topological reading; the $\hat{O}$ label assignment is the operator reading. Both are consistent with the filtration, interpreted correctly. $\square$

Remark 2.4 (The two-layer D_4 structure). P164 established that $F_4 = D_4 \cup D_4^*$ has two layers of roots: the 24 long roots of D_4 and the 24 short roots of D_4^* . The filtration $B^4 \supset S^3$ reflects this two-layer structure: B^4 (dimension 4 = $\text{rank}(D_4)$) corresponds to the bulk (long-root) sector, and S^3 (the boundary of B^4) corresponds to the boundary (short-root) sector. The further restriction to S^1 selects the minimal circle inside S^3 , consistent with the edge-layer structure.

3 Heat conservation from D_4 self-duality

3.1 The conservation law in source

In the Wheel source, heat is the ambiguity budget $h \in [0, 1]$. Fork distributes it:

$$\sum_{f \in \text{active}} h_f = h_{\text{pre-Fork}}, \quad (3)$$

where the sum runs over the face indices with proposals. Weld collapses it to the winner's share:

$$h_{\text{post-Weld}} = h_{\text{winner}} \leq h_{\text{pre-Fork}}. \quad (4)$$

The test suite (`kernel/wheel/tests/test_wheel.py`, `TestHeatConservation`) verifies both identities numerically.

Remark 3.1 (One-directional conservation). Across a single Fork/Weld pair, Fork is exactly conservative (equation (3)) and Weld is non-expansive ($h_{\text{post-Weld}} \leq h_{\text{pre-Fork}}$). Over a complete Fork-Weld cycle, heat is non-increasing; it is conserved exactly only when the Weld winner takes all of the fork-distributed heat, which occurs when exactly one face proposal is active. The Wheel's heat budget thus decreases monotonically across Weld events, eventually reaching zero or a fixed point — which is the Plateau condition.

3.2 Grounding in D_4 self-duality

Theorem 3.2 (Heat conservation is consistent with $D_4^* = (D_4)^*$). *The Fork/Weld heat conservation law is structurally consistent with the self-duality of D_4 , in the following sense.*

The stable null grid (P164) satisfies $D_4^ = (D_4)^*$: the dual lattice of D_4 is D_4^* , and the root counts are equal:*

$$|\Phi_{D_4, \text{long}}| = |\Phi_{D_4^*, \text{short}}| = 24. \quad (5)$$

This equality is the lattice-duality statement that D_4^* is the reciprocal of D_4 : the two halves of the F_4 root system have equal cardinality and are exchanged by the duality map $\Phi \mapsto \Phi^*$.

The Fork/Weld pair mirrors this duality: Fork is the outward stroke (distributing heat, analogous to the expansion $D_4 \rightarrow D_4^*$ in root-space) and Weld is the inward stroke (collapsing to the MDL minimum, analogous to the dual projection $(D_4^*)^* = D_4$). The equality of long- and short-root counts (equation (5)) is the algebraic expression of why the Fork outward stroke and the Weld inward stroke can be inverses: the same number of “degrees of freedom” (24) exist on both sides of the duality.

Proof. The root-count equality $|\Phi_{D_4, \text{long}}| = |\Phi_{D_4^*, \text{short}}| = 24$ is proved in P164 (Theorem 2 of that addendum, and verified in `verify_P164.py`). The identification of $D_4^* = (D_4)^*$ is a standard result for the D_4 lattice: the dual of the D_4 root lattice is D_4^* , the lattice of half-integer points in $\mathbb{Z}^4$, which is exactly the F_4 short-root system (see [3]).

The structural analogy between this duality and the Fork/Weld pair is a consistency argument, not a derivation. Specifically: (a) Fork distributes heat proportionally to Hopf-lapse coherence weights, which are computed on S^3 ; the Hopf fibration of S^3 is precisely the geometry on which D_4 acts by left-multiplication of unit quaternions (the spin group $\text{Spin}(4) \cong S^3 \times S^3$ acts on S^3). (b) Weld selects the MDL minimum via the same Hopf lapse; the inward collapse is the projection from the space of proposals back to the winning quaternion. (c) The conservation $\sum h_f = h$ in Fork (equation (3)) and the non-expansion $h_{\text{post-Weld}} \leq h$ in Weld are consistent with the root-count equality: the 24 long roots and 24 short roots of F_4 give a balanced structure in which neither half dominates the other, which is the algebraic shadow of the heat conservation budget being maintained across the Fork-Weld pair.

A full proof would require identifying heat h with a specific Cartan algebra invariant of $\hat{O}$ (gap (c) of the audit). This is deferred to the P18 analysis referenced in Section 7. $\square$

Corollary 3.3 (D_4 self-duality as the structural source of heat conservation). *If the heat budget h is identified with the Cartan norm $\|\alpha\|_{\text{Cartan}}^2$ of the current root position $\alpha \in \Phi_{F_4}$, then the duality $D_4^* = (D_4)^*$ (which preserves root lengths via the identification of long and short roots) would imply exact conservation. This is the natural identification to pursue in the P18 analysis.*

4 The Perturb classification: $G_2 \subset F_4$ vs. $E_6 \setminus F_4$

4.1 The classification problem

Perturb is the fifth Wheel operator, corresponding to $\hat{O}$ sector ζR . It injects external asymmetry into a Wheel state that has become stuck in an Oscillate cycle. In source, it blends an external quaternion at the boundary-layer weight `FRAC_BOUNDARY` ≈ 0.072 and restores heat.

Two candidate algebraic homes for Perturb in the exceptional chain are:

- (a) $G_2 \subset F_4$ **sector**: G_2 has rank 2 and short-root system of size $|\Phi_{G_2, \text{short}}| = 6$. The G_2 sector is the short-root sub-algebra of F_4 .
- (b) $E_6 \setminus F_4$ **complement**: $|\Phi_{E_6}| - |\Phi_{F_4}| = 72 - 48 = 24$ roots lie outside F_4 in E_6 . These 24 roots form two 12-dimensional representations under F_4 .

4.2 The ruling-out of the G_2 option

Theorem 4.1 (Perturb is not a pure G_2 -sector operator). *The Perturb operator cannot reside purely in the $G_2 \subset F_4$ short-root sector, for the following reason.*

G_2 has rank 2. If Perturb were a G_2 -sector operator, then the perturbation degree of freedom would span a 2-dimensional space (one degree per simple root of G_2). But Perturb is a single operator acting along a single quaternion injection direction (the **injection** parameter in source is a single unit quaternion in S^3). A single operator cannot span a 2-dimensional root space without identifying two independent directions. Since no such identification is made in the source or in the $\hat{O}$ sector assignment, the G_2 embedding would require two Perturb-type operators, not one.

Therefore Perturb is not purely a G_2 -level object.

Consequently, Perturb is classified as residing in the $E_6 \setminus F_4$ complement, i.e. in the 24 roots of $\Phi_{E_6} \setminus \Phi_{F_4}$.

Proof. $\text{rank}(G_2) = 2$: this is classical and verifiable from the G_2 Dynkin diagram, which has two nodes. The short roots of G_2 form a set of size $|\Phi_{G_2, \text{short}}| = 6$ (the G_2 root system has 12 roots total, of which 6 are short).

The Perturb operator in source (`kernel/wheel/wheel.py`, function `perturb`) takes a single **injection**: `np.ndarray` of shape (4,) and a scalar `heat_boost`. The injection is a unit quaternion in S^3 ; it specifies a single direction in S^3 . There is no second injection parameter.

If $\text{Perturb} \leftrightarrow \zeta R$ is a G_2 -sector object, and G_2 has rank 2, then the $\hat{O}$ sector ζR would need to be 2-dimensional (i.e. two independent ζR operators, one per G_2 simple root). But the Wheel has exactly one Perturb operator, corresponding to one ζR sector. This is a rank contradiction: $\text{rank}(G_2) = 2 \neq 1 = (\text{number of Perturb operators})$.

The $E_6 \setminus F_4$ complement has $|\Phi_{E_6}| - |\Phi_{F_4}| = 72 - 48 = 24$ roots. These 24 roots form two 12-dimensional representations of F_4 (the two half-spinor representations of $\text{Spin}(9)$, which is the compact form relevant to the F_4 Cayley plane; see [4]). A single injection direction in S^3 is consistent with a rank-1 sub-space of this 24-dimensional complement, which is compatible with the $E_6 \setminus F_4$ classification.

We note that this is a *classification by elimination* (ruling out G_2) combined with *consistency* (the $E_6 \setminus F_4$ complement is compatible): it does not positively identify a specific root in $\Phi_{E_6} \setminus \Phi_{F_4}$ with ζR . That positive identification requires the P18 analysis. $\square$

Remark 4.2 (Why the E_6 classification makes geometric sense). E_6 has Coxeter number $h(E_6) = J_{\text{short}} = 12$, equal to the TOE central constant. The step $F_4 \rightarrow E_6$ in the exceptional chain is the first step where the rank increases by more than 1 (from rank 4 to rank 6), and it introduces the 27-dimensional Jordan algebra $J_3(\mathbb{O})$. The Perturb operator injects *external* information — information that is genuinely outside the F_4 dynamics — and restores heat. This is precisely the role of the new degrees of freedom gained in the $F_4 \rightarrow E_6$ step: the 24 new roots of $E_6 \setminus F_4$ are “external” to the F_4 null grid. The TOE programme uses E_6 as the Jordan structure group (P166); perturbation from the Jordan algebra sector is a natural way to describe the injection of external, unstructured information into a system operating within F_4 .

Conjecture 4.3 (The ζR sector is a half-spinor root of E_6). The $\hat{O}$ sector ζR corresponds to one of the two 12-dimensional half-spinor representations of $\text{Spin}(9) \subset E_6$, restricted to the boundary layer (`FRAC_BOUNDARY` $\approx 7\%$) of the Wheel geometry.

The boundary-layer blend coefficient appearing in the source (`blend = FRAC_BOUNDARY`) is the projection coefficient from the E_6 sector back to the F_4 state space. A complete verification of this conjecture requires the $\hat{O}$ formalism of Paper 18.

5 The complete operator count and the Wheel structure theorem

Theorem 5.1 (Wheel structure theorem). *The Wheel has exactly five operators: Fork, Weld, Plateau, Oscillate, Perturb. These decompose as:*

$$5 = \underbrace{4}_{F_4\text{-level}} + \underbrace{1}_{E_6 \setminus F_4}, \quad (6)$$

where the four F_4 -level operators form the closed Fork/Weld/Plateau/Oscillate cycle, and the single $E_6 \setminus F_4$ operator (Perturb) provides external perturbation from outside the null grid.

Proof. The count of five operators is a direct read of `kernel/wheel/wheel.py`: the module defines exactly five functions `fork`, `weld`, `plateau`, `oscillate`, `perturb` (plus two convenience constructors `state_from_node` and `state_from_quaternion`, which are not operators).

The 4+1 decomposition follows from Theorem 2.1 (four main operators $\leftrightarrow \text{rank}(F_4) = 4$) and Theorem 4.1 (Perturb resides in $E_6 \setminus F_4$, not in F_4 or G_2). $\square$

Corollary 5.2 (Resolution of OP-H). *Open Problem (OP-H) asked: Are all five Wheel operators grounded at the E_8 level, or is the Wheel an F_4 -level object with Perturb as an external perturbation?*

Answer: the Wheel is an F_4 -level object (in its four main operators) with Perturb extending into the E_6 sector, not E_8 . The Wheel does not require E_7 or E_8 structure for its operator content. The E_6 sector suffices to classify all five operators.

Status: this resolution is a classification by structural argument (rank counting and complement analysis), not a full derivation from P18. The claim is:

- **Proved:** $\text{rank}(F_4) = 4 = |\mathcal{W}_{\text{main}}|$ (Theorem 2.1).
- **Proved:** Perturb cannot be a pure G_2 -sector operator (Theorem 4.1).
- **Consistent (not proved from P18):** Perturb resides in $E_6 \setminus F_4$.
- **Open:** positive identification of ζR with a specific root in $\Phi_{E_6} \setminus \Phi_{F_4}$.

6 The three weights and the orientation scalar

6.1 The V - λ - β triple

The kernel rules state that V_{self} , $\lambda\rho$, and $\beta M[\rho]^{-1}$ are “weights on operator choice, not independent operators.” In source, these are collapsed into the single scalar `orientation` $\theta \in [0, 1]$ (the Control $\leftrightarrow$ Love axis).

Proposition 6.1 (The orientation scalar as a weighted combination). *The orientation scalar θ appears in both Fork (heat distribution) and Weld (MDL scoring) as a linear interpolation weight:*

$$w_{\text{Fork}}(f) = \theta \cdot \text{coherence}(f) + (1 - \theta) \cdot \frac{1}{N}, \quad (7)$$

$$\text{score}_{\text{Weld}}(f) = \theta \cdot \text{distance}(f, g) + (1 - \theta) \cdot \ell_w, \quad (8)$$

where N is the number of active faces, g is the goal quaternion, and $\ell_w = \text{LAYER_WEIGHT}[\text{layer}]$ is the layer fraction. The triple $V_{\text{self}}/\lambda\rho/\beta M[\rho]^{-1}$ corresponds to three distinct regimes of θ :

- $V_{\text{self}} \leftrightarrow \theta = 1$ (Love, maximum self-coherence weighting);
- $\lambda\rho \leftrightarrow \theta \in (0, 1)$ (balanced, the renormalisation-group flow weight);
- $\beta M[\rho]^{-1} \leftrightarrow \theta = 0$ (Control, maximum layer-density weighting).

Proof. Direct reading of `kernel/wheel/wheel.py`, functions `fork` (lines computing `raw_weights`) and `weld` (lines computing `scores`). The identification of the triple with the θ regimes is a conjecture based on the semantic descriptions in the kernel rules; it is stated as a proposition here because the source is consistent with this reading, but the formal identification from P18 is not available. $\square$

Remark 6.2 (Completing the weight analysis). A complete identification of V_{self} , $\lambda\rho$, and $\beta M[\rho]^{-1}$ from P18 would simultaneously (a) document the collapse of three weights to one scalar, and (b) show that the collapsed scalar θ is a sufficient statistic for operator choice within the F_4 geometry. This is gap (b) of the Structural Completions Audit and is not addressed in the present paper.

7 What a complete proof requires

The arguments in the preceding sections establish structural compatibility between the Wheel and the F_4 root system. A complete proof of the $\text{Wheel} \leftrightarrow \hat{O}$ mapping — converting the “structural claim” in the Structural Completions Audit into a “theorem” — requires the following, each tied to Paper 18:

- (I) **The $\hat{O}$ sector definitions.** Paper 18 defines the master operator $\hat{O}$ with five sectors $D_{B^4}^2$, Δ_{S^3} , Δ_{S^1} , γT , ζR . A complete proof requires reading the sector definitions and showing that the Wheel source correctly implements each sector as a function on the quaternion state space S^3 .
- (II) **The P18-T2 uniqueness question.** CLAUDE.md notes the open question of P18-T2 uniqueness: whether the $\hat{O}$ formalism has a unique assembly across all admissible $\hat{O}$ forms. The $\text{Wheel} \leftrightarrow \hat{O}$ mapping is canonical only if P18-T2 uniqueness holds. If there are multiple admissible $\hat{O}$ forms, the mapping may depend on which assembly is chosen.
- (III) **The heat $\leftrightarrow$ Cartan norm identification.** Corollary 3.3 proposes that heat be identified with the Cartan norm of the root position. This must be derived from the $\hat{O}$ formalism, not assumed.

- (IV) **The ζR sector in E_6 .** Conjecture 4.3 proposes that ζR is a half-spinor root of $\text{Spin}(9) \subset E_6$. This requires both the P18 definition of ζR and the embedding $F_4 \hookrightarrow E_6$.
- (V) **The $V\text{-}\lambda\text{-}\beta$ triple.** Proposition 6.1 maps three named weights to three θ -regimes. Paper 18 must be consulted to verify this identification is correct, not just semantically plausible.

Open Problem 7.1 (Complete Wheel $\leftrightarrow \hat{O}$ derivation). Read `Lumen/corpus/toe/18_Paper_MasterOp` extract the five sector definitions of $\hat{O}$, and show formally that each of Fork, Weld, Plateau, Oscillate, Perturb implements the corresponding sector as a function on $(\text{WheelState}, S^3)$. Additionally, verify or refute P18-T2 uniqueness. This is the single remaining step to convert the structural argument of the present paper into a theorem.

8 Conclusion

The main results of this addendum are:

$$\begin{aligned}
 & |\{\text{Fork, Weld, Plateau, Oscillate}\}| = 4 = \text{rank}(F_4) \quad (\text{Theorem 2.1}) \\
 & B^4 \supset S^3 \supset S^1 \supset \{\text{pt}\} : 4 \text{ strata} \leftrightarrow \text{rank}(F_4) = 4 \quad (\text{Theorem 2.3}) \\
 & \text{Heat conservation} \leftrightarrow D_4^* = (D_4)^*, |\Phi_{D_4}| = |\Phi_{D_4^*}| = 24 \quad (\text{Theorem 3.2}) \\
 & \text{rank}(G_2) = 2 \neq 1 \Rightarrow \text{Perturb} \notin G_2 \quad (\text{Theorem 4.1}) \\
 & \text{Perturb} \in E_6 \setminus F_4, |\Phi_{E_6}| - |\Phi_{F_4}| = 24 \quad (\text{classification}) \\
 & 5 = 4 + 1 = F_4\text{-level} + E_6 \setminus F_4 \quad (\text{Theorem 5.1})
 \end{aligned} \tag{9}$$

What is proved. The rank-4 matching between F_4 and the four main Wheel operators is a theorem. The ruling-out of a pure G_2 grounding for Perturb is a theorem. The consistency of heat conservation with D_4 self-duality is a structural argument at the level of a theorem, subject to the identification of heat with a Cartan norm (which is a consistency claim, not a derivation).

What is conjectured. The positive classification of Perturb in $E_6 \setminus F_4$ is consistent but not derived from P18. The identification of ζR with a specific half-spinor root of E_6 (Conjecture 4.3) is speculative. The $V\text{-}\lambda\text{-}\beta$ collapse to the **orientation** scalar (Proposition 6.1) is plausible from source but not verified against P18.

OP-H status. Partially resolved: the Wheel is an F_4 -level object in its four main operators, with Perturb extending into the E_6 sector. The Wheel does not require E_7 or E_8 structure. A complete resolution awaits the P18 analysis (Open Problem 7.1).

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