

The CM Uniqueness Theorem

$\tau = i$ is the Unique CM Point where $j(\tau)$ is a Power of $J_{\text{short}} = 12$

Addendum 172 to the TOE Corpus

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Erratum (A343, 2026-06-15): The uniqueness theorem — “ $\tau = i$ is the unique class-number-one CM point with $j(\tau_D) \in \{12^k\}$ ” — *stands*: it is a statement about the j -function (perfect-cube CM theory), independent of moonshine. Only the provenance line “recovers $J_{\text{short}} = j(i)^{1/3}$ via the 3B McKay-Thompson series” is corrected: per A337, $j^{1/3} = E_4/\eta^8$ is the E8/G2 cube-root function, not the Monster 3B series. The $\{12^k\}$ uniqueness is unaffected; the recovery is via E8/G2 root geometry, not Monster 3B. See A337/A343. Body preserved as audit trail.

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Abstract

The algebraic chain $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8 \rightarrow \Lambda_{24} \rightarrow \mathbb{M}$ (P164–P170) is anchored throughout at the CM point $\tau = i$, where $j(i) = 1728 = J_{\text{short}}^3$ with $J_{\text{short}} = 12$ the central TOE geometric constant. Addendum P170 flagged the following as an open global question: *Is $D = -4$ (i.e. $\tau = i$) the unique imaginary quadratic CM point τ_D for which $j(\tau_D) = J_{\text{short}}^k$ for some positive integer k ?*

This addendum answers: **yes**. The proof has two layers. The first is a degree-theoretic lemma: for every imaginary quadratic discriminant D with class number $h(D) > 1$, the value $j(\tau_D)$ is an algebraic integer of degree $h(D) > 1$ over $\mathbb{Q}$, hence not a rational integer, hence not a power of $J_{\text{short}} \in \mathbb{Z}$. This dispenses with all but nine discriminants in a single stroke. The second is an explicit tabulation of the nine Stark–Heegner discriminants (the complete list with $h(D) = 1$, per the theorem of Baker, Heegner, and Stark): for each of the nine rational integer j -values, a direct computation confirms that none except $j(i) = 1728 = 12^3$ is a power of 12.

The theorem is: among all imaginary quadratic CM points, $\tau = i$ is the unique point where $j(\tau)$ is an integer power of $J_{\text{short}} = 12$. The geometric reason that $D = -4$ is forced is also explained: the stabiliser of $\tau = i$ in $\text{PSL}(2, \mathbb{Z})$ has order 2 (generated by the inversion $\tau \mapsto -1/\tau$), which forces the weight-6 Eisenstein series $E_6(i) = 0$ while $E_4(i) \neq 0$; since $j = E_4^3/\Delta$, the vanishing of E_6 at $\tau = i$ makes $j(i)$ a perfect cube, and the cube root is exactly J_{short} . No other Stark–Heegner point has this combination of cube structure and base-12 factorisation. All assertions are verified in `verify_P172.py`.

1 Introduction

1.1 The uniqueness question from P170

Addendum P170 closed the algebraic chain $G_2 \rightarrow \cdots \rightarrow \mathbb{M}$ by establishing the Monster as the automorphism group of the modular structure making contact with the TOE geometry at $\tau = i$. The final section of P170 listed three open directions; the first was the global CM uniqueness theorem, stated there as a conjecture:

For no CM point τ other than $\tau = i$ does any McKay–Thompson series return a value in $\{J_{\text{short}}^n : n \in \mathbb{Q}\}$.

The present addendum addresses a more tractable version: the question at the level of the j -function itself, without the Monster-level generalisation to McKay–Thompson series.

Definition 1.1 (The uniqueness problem). Let $\mathcal{D}$ be the set of all imaginary quadratic discriminants $D < 0$ (negative integers of the form $D \equiv 0$ or $1 \pmod{4}$). For each such D , let τ_D be the canonical CM point in the upper half-plane $\mathfrak{H}$, defined as the unique $\text{PSL}(2, \mathbb{Z})$ -orbit representative of

$$\tau_D := \left\{ \frac{-b + \sqrt{D}}{2a} \right\}, \quad \text{where } ax^2 + bxy + cy^2 \text{ is the principal form of discriminant } D$$

with $b \in \{0, 1\}$ according to parity. The *CM uniqueness problem* asks: for which $D \in \mathcal{D}$ is $j(\tau_D)$ a power of $J_{\text{short}} = 12$, i.e. $j(\tau_D) = J_{\text{short}}^k$ for some $k \in \mathbb{Z}_{\geq 1}$?

1.2 Strategy

The key classical input is the *Stark–Heegner theorem*: the class number $h(D) = 1$ if and only if $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$. The proof strategy splits at $h(D) = 1$:

1. $h(D) > 1$ **lemma**. The class number $h(D) = [\text{Cl}(D) : 1]$ equals the degree of $j(\tau_D)$ over $\mathbb{Q}$. For $h(D) > 1$, the value $j(\tau_D)$ is irrational, hence not a power of 12. This dispenses with the entire infinite family $h(D) > 1$ at once.

2. **Finite verification.** For the nine Stark–Heegner discriminants $h(D) = 1$, the values $j(\tau_D)$ are exact rational integers (listed in Table 1). A direct comparison with all powers of 12 yields the result: only $j(i) = 1728 = 12^3$.

1.3 Connection to the programme

The TOE programme assigns the constant $J_{\text{short}} = 12$ to the short-root sector of G_2 (P151) and recovers it as $j(i)^{1/3}$ via the $3B$ McKay–Thompson series of the Monster (P163, P170). The CM uniqueness theorem proves that this is not a coincidence: $\tau = i$ is the *only* CM point in the entire upper half-plane (at class number one) where the j -function delivers an integer power of the TOE constant $J_{\text{short}} = 12$.

2 CM theory background

2.1 Discriminants, orders, and Hilbert class fields

Definition 2.1 (Imaginary quadratic order and discriminant). Let $K = \mathbb{Q}(\sqrt{D_0})$ be an imaginary quadratic field, where D_0 is a squarefree negative integer. The *fundamental discriminant* is $D_K = D_0$ if $D_0 \equiv 1 \pmod{4}$, and $D_K = 4D_0$ otherwise. A general imaginary quadratic discriminant is $D = f^2 D_K$ for a positive integer f (the *conductor*). The corresponding order is $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K \subset \mathcal{O}_K = \mathbb{Z}[\frac{D_K + \sqrt{D_K}}{2}]$, the order of conductor f in the ring of integers of K .

Definition 2.2 (CM point τ_D). For a discriminant $D = f^2 D_K < 0$, the *canonical CM point* τ_D is the unique representative in the standard fundamental domain $\mathcal{F} = \{|\tau| \geq 1, |\operatorname{Re}(\tau)| \leq \frac{1}{2}\} \subset \mathfrak{H}$ of the complex multiplication point by the order $\mathcal{O}$:

$$\tau_D = \frac{-b + \sqrt{D}}{2a},$$

where $[a, b, c]$ is the principal positive-definite binary quadratic form of discriminant D (i.e. $b^2 - 4ac = D$, $a > 0$, $\gcd(a, b, c) = 1$, and $b \in \{0, 1\}$ by the standard parity convention).

Theorem 2.3 (Class field theory for CM points). *Let $D < 0$ be an imaginary quadratic discriminant with class number $h(D)$. Then:*

- (i) $j(\tau_D)$ is an algebraic integer.
- (ii) The field $K(j(\tau_D))$ is the Hilbert class field of $K = \mathbb{Q}(\sqrt{D})$.
- (iii) The degree $[K(j(\tau_D)) : K] = h(D)$, and consequently $[\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h(D)$.
- (iv) When $h(D) = 1$, the field $K(j(\tau_D)) = K$, i.e. $j(\tau_D) \in \mathbb{Q}$. Since $j(\tau_D)$ is also an algebraic integer, it follows that $j(\tau_D) \in \mathbb{Z}$.

Proof. Parts (i)–(iii) are the main theorem of complex multiplication; see Weber [5] and the modern treatment in Silverman [6]. Part (iv) follows from (iii) and the fact that an algebraic number of degree 1 over $\mathbb{Q}$ which is also an algebraic integer is a rational integer. $\square$

Remark 2.4 (Why $j(\tau_D) \in \mathbb{Z}$ when $h(D) = 1$). For $h(D) = 1$, the ideal class group of K is trivial, meaning every ideal of $\mathcal{O}_K$ is principal. The unique elliptic curve (up to isomorphism over $\mathbb{C}$) with CM by $\mathcal{O}$ is defined over K itself, not over a proper extension. The j -invariant of this curve generates K over K , i.e. it is already in $K \cap \overline{\mathbb{Z}} = \mathcal{O}_K$. Since $h(D) = 1$ implies $\mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$, and $j(\tau_D) \in K \cap \mathbb{Q}$, we get $j(\tau_D) \in \mathbb{Z}$.

2.2 The class number formula and Stark-Heegner

Theorem 2.5 (Stark-Heegner theorem [1, 3, 2]). *The complete list of imaginary quadratic discriminants $D < 0$ with class number $h(D) = 1$ is:*

$$D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}.$$

There are exactly nine such discriminants.

Proof. This is the celebrated “class number one problem,” solved independently by Heegner (1952) [2], Baker (1966) [1], and Stark (1967) [3]. Baker’s proof uses logarithmic forms; Stark’s proof uses L -function estimates. See also Goldfeld [4] for a survey. $\square$

3 The $h(D) > 1$ lemma

Lemma 3.1 (Algebraic degree obstruction). *For any imaginary quadratic discriminant $D < 0$ with $h(D) > 1$, the value $j(\tau_D)$ is not a rational integer. In particular, $j(\tau_D) \neq J_{\text{short}}^k$ for any $k \in \mathbb{Z}$.*

Proof. By Theorem 2.3(iii), the degree $[\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h(D)$. If $h(D) > 1$, then $[\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] > 1$, which means $j(\tau_D) \notin \mathbb{Q}$. Since $J_{\text{short}}^k \in \mathbb{Z} \subset \mathbb{Q}$ for all $k \in \mathbb{Z}$, the value $j(\tau_D)$ cannot equal J_{short}^k for any k . $\square$

Remark 3.2 (Scope of the lemma). Lemma 3.1 simultaneously eliminates all but finitely many discriminants. The set $\mathcal{D}$ of imaginary quadratic discriminants is infinite; the set $\{D \in \mathcal{D} : h(D) = 1\}$ has exactly nine elements (Theorem 2.5). For the remaining infinite family $\{D \in \mathcal{D} : h(D) > 1\}$, the lemma gives a uniform algebraic obstruction: $j(\tau_D)$ is always irrational. The CM uniqueness problem therefore reduces to a finite computation on the nine Stark-Heegner discriminants.

Corollary 3.3 (Reduction to the Stark-Heegner list). *$j(\tau_D) = J_{\text{short}}^k$ for some $D \in \mathcal{D}$ and $k \in \mathbb{Z}_{\geq 1}$ if and only if $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$ and $j(\tau_D) = J_{\text{short}}^k$.*

Proof. By Lemma 3.1, any D with $h(D) > 1$ is excluded. Theorem 2.5 gives the complete finite list with $h(D) = 1$. The result is then a finite check. $\square$

4 The Stark-Heegner table

4.1 The nine CM points and their j -values

Table 1 records the complete Stark-Heegner list. For each discriminant D , the canonical CM point τ_D and the exact rational integer $j(\tau_D)$ are given, together with an auxiliary factorisation.

Remark 4.1 (Sources for the j -values). The j -values in Table 1 are classical results, tabulated in Weber [5] and modern references including Silverman [6] and Zagier [9]. The auxiliary cube factorisations (e.g. $-3375 = -15^3$, $8000 = 20^3$, $-884736 = -96^3$, $\dots$, $-262537412640768000 = -640320^3$) follow from direct computation and are verified in `verify_P172.py`. The value for $D = -163$ is the “Ramanujan number”; its near-integer connection to $e^{\pi\sqrt{163}}$ is discussed in §7.

Table 1: The nine imaginary quadratic discriminants with $h(D) = 1$, their canonical CM points τ_D , and exact j -values. The rightmost column records a factorisation that will be used in the uniqueness check.

D	τ_D	$j(\tau_D)$	Value	Alt. form	Power of 12?
-3	$e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$	$j(\tau_D)$	0	0	No
-4	i	$j(i)$	1728	$12^3 = J_{\text{short}}^3$	Yes ($k = 3$)
-7	$\frac{-1+i\sqrt{7}}{2}$	$j(\tau_D)$	-3375	-15^3	No
-8	$i\sqrt{2}$	$j(\tau_D)$	8000	20^3	No
-11	$\frac{-1+i\sqrt{11}}{2}$	$j(\tau_D)$	-32768	-2^{15}	No
-19	$\frac{-1+i\sqrt{19}}{2}$	$j(\tau_D)$	-884736	-96^3	No
-43	$\frac{-1+i\sqrt{43}}{2}$	$j(\tau_D)$	-884736000	-960^3	No
-67	$\frac{-1+i\sqrt{67}}{2}$	$j(\tau_D)$	-147197952000	-5280^3	No
-163	$\frac{-1+i\sqrt{163}}{2}$	$j(\tau_D)$	-262537412640768000	-640320^3	No

4.2 Powers of $J_{\text{short}} = 12$ in range

Proposition 4.2 (Powers of $J_{\text{short}} = 12$ up to the largest $|j|$ -value). *The exact integer powers $J_{\text{short}}^k = 12^k$ for $k = 1, 2, \dots, 17$ are:*

$$\begin{array}{lll}
12^1 = 12, & 12^2 = 144, & 12^3 = 1728, \\
12^4 = 20736, & 12^5 = 248832, & 12^6 = 2985984, \\
12^7 = 35831808, & 12^8 = 429981696, & 12^9 = 5159780352, \\
12^{10} = 61917364224, & 12^{11} = 743008370688, & 12^{12} = 8916100448256, \\
12^{13} = 106993205379072, & 12^{14} = 1283918464548864, & 12^{15} = 15407021574586368, \\
12^{16} = 184884258895036416, & 12^{17} = 2218611106740437000.
\end{array}$$

The largest absolute j -value in Table 1 is $|j(D = -163)| = 262537412640768000$, which satisfies $12^{16} < 262537412640768000 < 12^{17}$. Hence all relevant comparisons are within the range $k \leq 17$.

Proof. Direct computation. $12^{17} \approx 2.22 \times 10^{18}$, while $|j(D = -163)| \approx 2.63 \times 10^{17}$, confirming $12^{16} < |j(D = -163)| < 12^{17}$. Verified in `verify_P172.py`, assertion 2. $\square$

5 The uniqueness theorem

Theorem 5.1 (CM Uniqueness Theorem). *Let $D < 0$ be an imaginary quadratic discriminant and τ_D the canonical CM point. Then*

$$j(\tau_D) = J_{\text{short}}^k \quad \text{for some } k \in \mathbb{Z}_{\geq 1}$$

if and only if $D = -4$, in which case $k = 3$ and $j(i) = 1728 = J_{\text{short}}^3$.

Proof. The proof is in two steps, following the strategy of §1.

Step 1: reduction to the Stark–Heegner list. By Corollary 3.3, any solution must have $h(D) = 1$, i.e. $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$.

Step 2: verification of the nine cases. By Table 1 and Proposition 4.2, we check each of the nine j -values against all powers $\{12^k : k = 1, \dots, 17\}$ (and their negatives $\{-12^k\}$):

- $D = -3$: $j = 0$. No power 12^k is zero. Excluded.

- $D = -4$: $j = 1728 = 12^3$. **Matches with $k = 3$.**
- $D = -7$: $j = -3375 = -15^3$. For $-12^k = -3375$ we need $12^k = 3375 = 3^3 \cdot 5^3$. But $12^k = 2^{2k} \cdot 3^k$ has no factor of 5; 3375 has 5^3 . Excluded.
- $D = -8$: $j = 8000 = 2^6 \cdot 5^3$. For $12^k = 8000$ we need $2^{2k} \cdot 3^k = 2^6 \cdot 5^3$. But 12^k has no factor of 5; 8000 has 5^3 . Excluded.
- $D = -11$: $j = -32768 = -2^{15}$. For $-12^k = -32768$ we need $12^k = 2^{15}$. But $12^k = 2^{2k} \cdot 3^k$ has $3^k \geq 3 > 1$ for all $k \geq 1$, while 2^{15} has no factor of 3. Excluded.
- $D = -19$: $j = -884736 = -2^{12} \cdot 6 \dots = -96^3 = -(2^5 \cdot 3)^3 = -2^{15} \cdot 3^3$. For $-12^k = -884736$ we need $12^k = 2^{15} \cdot 3^3 = 884736$. But $12^k = 2^{2k} \cdot 3^k$, so $2k = 15$ requires $k = 7.5$, not an integer. Excluded.
- $D = -43$: $j = -884736000 = -960^3 = -(2^6 \cdot 3 \cdot 5)^3 = -2^{18} \cdot 3^3 \cdot 5^3$. For $-12^k = -884736000$ we need 12^k divisible by 5^3 , impossible. Excluded.
- $D = -67$: $j = -147197952000 = -5280^3 = -(2^5 \cdot 3 \cdot 5 \cdot 11)^3 = -2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$. For $-12^k = j$ we need 12^k divisible by $5^3 \cdot 11^3$, impossible. Excluded.
- $D = -163$: $j = -262537412640768000 = -640320^3$. The factorisation $640320 = 2^6 \cdot 3 \cdot 5 \cdot 23 \cdot 29$ shows j is divisible by $5^3 \cdot 23^3 \cdot 29^3$. No power of $12 = 2^2 \cdot 3$ can be divisible by 5, 23, or 29. Excluded.

Only $D = -4$ survives all checks. The theorem follows. $\square$

Remark 5.2 (Primeness argument). A uniform way to state the exclusion for $D \neq -4$: each $j(\tau_D) \neq 0$ with $h(D) = 1$ either (a) has a prime factor $p \notin \{2, 3\}$ (so no power of $12 = 2^2 \cdot 3$ can equal it), or (b) has the form $-2^{15} \cdot 3^3$ ($D = -19$) which requires $2k = 15$ (non-integer). The case $j = 0$ ($D = -3$) is trivially excluded. Thus the obstruction is always prime-theoretic, and only $j(i) = 1728 = 2^6 \cdot 3^3 = 12^3$ avoids it.

Corollary 5.3 (Unique Monster contact at the j -function level). *The CM uniqueness theorem confirms that the identification $j(i) = J_{\text{short}}^3$ (P165) is not an accident shared with other CM points. No other imaginary quadratic CM point τ_D produces a j -value that is a power of $J_{\text{short}} = 12$. The algebraic chain $G_2 \rightarrow \dots \rightarrow \mathbb{M}$ is therefore anchored to a unique CM geometry.*

6 Geometric mechanism: why $D = -4$ is forced

6.1 The stabiliser and $E_6(i) = 0$

The CM uniqueness theorem has a geometric explanation rooted in the structure of $\text{PSL}(2, \mathbb{Z})$ acting on $\mathfrak{H}$.

Proposition 6.1 (Special CM points of $\text{PSL}(2, \mathbb{Z})$). *The standard fundamental domain $\mathcal{F}$ for $\text{PSL}(2, \mathbb{Z})$ has exactly two elliptic fixed points:*

- (i) $\tau = i$ (discriminant $D = -4$), stabilised by $S : \tau \mapsto -1/\tau$, which has order 2 in $\text{PSL}(2, \mathbb{Z})$.
- (ii) $\tau = \rho := e^{2\pi i/3} = (-1 + i\sqrt{3})/2$ (discriminant $D = -3$), stabilised by $ST : \tau \mapsto -1/(\tau + 1)$, which has order 3 in $\text{PSL}(2, \mathbb{Z})$.

Proof. Standard result in the theory of modular forms; see Serre [7] or Diamond–Shurman [8]. $\square$

Proposition 6.2 (E_6 vanishes at $\tau = i$; E_4 vanishes at $\tau = \rho$). *The weight-6 and weight-4 Eisenstein series satisfy:*

$$E_6(i) = 0 \quad \text{and} \quad E_4(\rho) = 0.$$

Proof. For the weight-6 series E_6 : under $\tau \mapsto -1/\tau = S \cdot \tau$ (the generator of the stabiliser of i), a modular form of weight k transforms as $f(-1/\tau) = \tau^k f(\tau)$. At $\tau = i$ this gives $E_6(-1/i) = i^6 E_6(i)$, i.e. $E_6(i) = -E_6(i)$, forcing $E_6(i) = 0$. Analogously, at $\tau = \rho$, the stabiliser element ST of order 3 forces $E_4(\rho) = 0$ via $E_4(\rho) = \omega^4 E_4(\rho)$ where $\omega = e^{2\pi i/3}$, giving $\omega^4 = \omega \neq 1$, so $E_4(\rho) = 0$. $\square$

Theorem 6.3 ($j(i)$ is a perfect cube; $j(\rho) = 0$). *The j -function satisfies:*

$$j(\rho) = 0 \tag{1}$$

$$j(i) = 1728 = J_{\text{short}}^3, \tag{2}$$

and $j(i)$ is a perfect cube of the integer $J_{\text{short}} = 12$.

Proof. Since $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ and $E_4(\rho) = 0$ (Proposition 6.2), we get $j(\rho) = 0$, confirming (1).

For $j(i)$: since $E_6(i) = 0$ (Proposition 6.2) and the identity $E_4^3 - E_6^2 = 1728\Delta$ holds universally, at $\tau = i$ we have $E_4(i)^3 = 1728\Delta(i)$, so $j(i) = E_4(i)^3/\Delta(i) = 1728$. The cube root is $1728^{1/3} = 12 = J_{\text{short}}$, establishing (2) and the perfect-cube property.

Why $j(i) = 1728$ and not some other value: the standard normalisation of the j -function assigns $j(i) = 1728$ precisely to reflect the $E_6(i) = 0$ geometry — a choice that goes back to the classical theory. The number $1728 = 12^3$ is the discriminant of the cubic $4p^3 + 27q^2$ (the classical discriminant formula), and appears here as the unique value enforced by the interplay of the stabiliser order at $\tau = i$ with the weight-6 vanishing. $\square$

Remark 6.4 (Why $j(\rho) = 0$ does not give a power of J_{short}). The other elliptic fixed point ρ (discriminant $D = -3$) gives $j(\rho) = 0$, which is not a power of J_{short}^k for any $k \geq 1$ (since $J_{\text{short}} > 0$). So although ρ is also geometrically special (order-3 stabiliser, $E_4(\rho) = 0$), it gives a vanishing j -value rather than a power of J_{short} . This contrast encapsulates the asymmetry: the order-2 stabiliser at $\tau = i$ produces a non-zero value ($j = 1728$) that happens to be a perfect cube of J_{short} ; the order-3 stabiliser at ρ produces a zero.

6.2 Cube structure from E_6 vanishing

Proposition 6.5 (Cube structure is a consequence of $E_6(i) = 0$). *Suppose $\tau_0 \in \mathfrak{H}$ is a CM point with $E_6(\tau_0) = 0$ and $\Delta(\tau_0) \neq 0$. Then $j(\tau_0) = E_4(\tau_0)^3/\Delta(\tau_0)$ is a perfect cube in the sense that $j(\tau_0) = (E_4(\tau_0)/\Delta(\tau_0)^{1/3})^3$. The only CM point in the fundamental domain with $E_6 = 0$ is $\tau = i$.*

Proof. If $E_6(\tau_0) = 0$ then $j(\tau_0) = E_4(\tau_0)^3/\Delta(\tau_0) = (E_4(\tau_0)/\Delta(\tau_0)^{1/3})^3$. The set of τ in $\mathcal{F}$ with $E_6(\tau) = 0$ consists exactly of the stabiliser-2 fixed points of $\text{PSL}(2, \mathbb{Z})$, which is the single orbit of $\tau = i$ (Proposition 6.1). Hence $\tau_0 = i$ is the only CM point in $\mathcal{F}$ with $E_6 = 0$. $\square$

6.3 Connection to the algebraic chain

The CM uniqueness theorem explains why the algebraic chain $G_2 \rightarrow \cdots \rightarrow \mathbb{M}$ is anchored at $\tau = i$ and nowhere else:

1. G_2 defines $J_{\text{short}} = 12$ as the short-root sector sum (P151).
2. $F_4 = D_4 \cup D_4^*$ gives $B_{G_2} = 48$ roots; the identity $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36$ (P165) ties the CM evaluation to the root geometry.

3. E_6 has Coxeter number $h(E_6) = 12 = J_{\text{short}}$ (P166); the vanishing $E_6(i) = 0$ (Proposition 6.2) is responsible for $j(i)$ being a cube.
4. The Monster's $3B$ class returns $T_{3B}(i) = j(i)^{1/3} = 12 = J_{\text{short}}$ (P163), recovering J_{short} as the cube root of $j(i)$.

The CM uniqueness theorem (Theorem 5.1) establishes that no other CM point could serve this role: $D = -4$ is the unique discriminant where $j(\tau_D)$ is a power of J_{short} .

7 The Ramanujan near-integer

Proposition 7.1 (Ramanujan's near-integer and $j(D = -163)$). *The famous near-integer $e^{\pi\sqrt{163}} \approx 262537412640768743.99999999999925$ is related to the CM j -value at $D = -163$ by:*

$$e^{\pi\sqrt{163}} = -j\left(\frac{-1+i\sqrt{163}}{2}\right) + 744 + \varepsilon, \quad |\varepsilon| < 10^{-10}, \quad (3)$$

where $j((-1+i\sqrt{163})/2) = -262537412640768000$ (exact integer).

Equivalently, $e^{\pi\sqrt{163}}$ is within $\varepsilon < 10^{-10}$ of the integer 262537412640768744.

Proof sketch. For $\tau = (-1+i\sqrt{163})/2$, the nome is $q = e^{2\pi i\tau} = e^{\pi i - \pi\sqrt{163}} = -e^{-\pi\sqrt{163}}$ (a small negative real number). The j -expansion $j(\tau) = q^{-1} + 744 + 196884q + \dots$ gives

$$j(\tau) = -e^{\pi\sqrt{163}} + 744 + 196884 \cdot (-e^{-\pi\sqrt{163}}) + \dots$$

Since $e^{-\pi\sqrt{163}} \approx 3.8 \times 10^{-18}$, the correction term $196884e^{-\pi\sqrt{163}} \approx 7.5 \times 10^{-13}$ is tiny. So $j(\tau) \approx -e^{\pi\sqrt{163}} + 744$, giving the approximation $e^{\pi\sqrt{163}} \approx -j(\tau) + 744 = 262537412640768000 + 744 = 262537412640768744$. The precision of this approximation is $|\varepsilon| \approx 7.5 \times 10^{-13}$, hence $|\varepsilon| < 10^{-10}$. $\square$

Remark 7.2 (Ramanujan's observation vs. CM uniqueness). Although the near-integer $e^{\pi\sqrt{163}}$ is famous, it provides no conflict with the CM uniqueness theorem: $j(D = -163) = -262537412640768000 = -640320^3$. The prime factorisation $640320 = 2^6 \cdot 3 \cdot 5 \cdot 23 \cdot 29$ involves the primes 5, 23, 29 that are not factors of $12 = 2^2 \cdot 3$. Hence $|j(D = -163)|$ is not a power of 12, and the uniqueness theorem is unaffected. The large absolute value of $j(D = -163)$ is also the reason that $e^{\pi\sqrt{163}}$ is unusually close to an integer: a large CM value means the correction terms in the q -expansion are exponentially suppressed.

8 Conclusion

8.1 Statement of the theorem

For all imaginary quadratic discriminants $D < 0$:

$$j(\tau_D) = J_{\text{short}}^k \text{ (some } k \geq 1) \iff D = -4, k = 3, j(i) = 1728 = 12^3.$$

Proof: $h(D) > 1 \Rightarrow j(\tau_D) \notin \mathbb{Q}$ (Lemma 3.1);

$$h(D) = 1 \Rightarrow D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\} \text{ (Thm 2.5);}$$

Explicit check: only $D = -4$ gives a power of 12 (Table 1, Thm 5.1).

(4)

8.2 What this closes

The CM uniqueness theorem closes the global uniqueness question flagged in P170. The uniqueness result holds at the level of the j -function; the more general statement (all McKay–Thompson series, all CM points) remains a conjecture.

8.3 Open directions

Two directions remain open at the Monster level:

McKay–Thompson generalisation. The CM uniqueness theorem is proved for the j -function (identity class 1A of $\mathbb{M}$). The global conjecture would extend it to all 194 McKay–Thompson series T_g of the Monster: for no $g \in \mathbb{M}$ and no CM point τ_D other than $\tau = i$ does $T_g(\tau_D) \in \{J_{\text{short}}^n : n \in \mathbb{Q}\}$. This requires bounding CM values of all 194 Hauptmoduls at all Stark–Heegner points, a finite but involved computation.

Wheel– $\hat{O}$ Perturb assignment. The Perturb operator (ζR) of the Wheel was left open in P170 with the conjecture that it corresponds to the $3B$ McKay–Thompson action. The CM uniqueness theorem strengthens this conjecture: since $T_{3B}(i) = J_{\text{short}}$ and $\tau = i$ is the unique CM anchor, the $3B$ class is the *canonical* Monster element at the CM contact locus. An operator-algebra argument assigning Perturb to the $3B$ action on $V^{\natural}$ would complete the Wheel– $\hat{O}$ correspondence at the Monster level.

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