

G_1 as a Modular Residual: The Fractional Gap between the TOE Breath Period and One Quarter of $j(i)$ Addendum 167 to the TOE Corpus

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Abstract

The G_1 gap was introduced as $G_1 = (432 - \Omega)/\Omega$, where $\Omega = \pi\alpha^{-1}$ is the TOE breath period and 432 is its rational approximation. Addendum 156 (P156) proved $G_1 \neq 0$ via Lindemann–Weierstrass, and proved that G_1 is unreachable by Haar averaging over any compact group (Schur obstruction).

The present addendum supplies a new structural identity: since $432 = j(i)/4 = J_{\text{short}}^3/4 = 1728/4$ (P158), the G_1 gap is precisely the fractional deviation of the breath period Ω from the modular value $j(i)/4$ at the CM point $\tau = i$:

$$G_1 = \frac{j(i)}{4\Omega} - 1 = \frac{J_{\text{short}}^3}{4\pi\alpha^{-1}} - 1.$$

We prove: (i) this is a tautological reformulation of the definition once P158 is applied, but it carries geometric content: G_1 measures exactly how far Ω falls from the “modular slot” $j(i)/4$; (ii) the breath period reconstructs as $\Omega = j(i)/(4(1 + G_1))$, expressing Ω as a G_1 -corrected quarter of $j(i)$; (iii) the absolute gap $j(i) - 4\Omega = 5.9510\dots$ is close to but strictly less than $J_{\text{short}}/2 = 6$, proved non-exact by transcendence; (iv) the polynomial identity $4\Omega = 16\pi^4 + 4\pi^3 + 4\pi^2 = (4\pi)\alpha^{-1}$ gives 4Ω the geometric interpretation of the surface area of the unit S^2 times α^{-1} ; (v) G_1 is transcendental; and (vi) no modular form evaluates to 4Ω at any CM point of discriminant $D = -4$, confirming that G_1 is an irreducible modular residual. All numerical claims are verified to 55 significant figures by `verify_G1_modular.py`.

1 Background and motivation

1.1 The G_1 gap and its prior status

The TOE breath period is the positive real number

$$\Omega = \pi\alpha^{-1}, \quad \alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \tag{1}$$

so that $\Omega = 4\pi^4 + \pi^3 + \pi^2$. Numerically, $\Omega \approx 430.512$. The integer 432 is the nearest integer to Ω that arises naturally in the TOE construction (see P 156 for the complete motivation). The G_1 gap is defined as the fractional deviation:

$$G_1 = \frac{432 - \Omega}{\Omega}. \tag{2}$$

Addendum 156 established three properties of G_1 :

1. *Non-zero*: Ω is transcendental by Lindemann–Weierstrass (since π is transcendental and α^{-1} is a nonconstant polynomial in π), while $432 \in \mathbb{Q}$, so $\Omega \neq 432$ and $G_1 \neq 0$.
2. *Irreducible*: The Schur obstruction shows that G_1 cannot be removed by Haar averaging over any compact group acting on the geometry of the breath period.
3. *Positive*: $\Omega < 432$, so $G_1 > 0$.

What was not addressed in P156 is the structural origin of the integer 432. That question is answered here.

1.2 The j -function at $\tau = i$ and the number 432

Addendum 158 (P158) derived, from the root geometry of G_2 and F_4 , the identity

$$j(i) = J_{\text{short}}^3 = 12^3 = 1728, \tag{3}$$

where $J_{\text{short}} = 12$ is the G_2 short-root sector sum (P151) and j is the Klein j -invariant. Additively, $j(i)/4 = 1728/4 = 432$. This is not a coincidence of notation: 432 enters the TOE as the integer nearest to Ω , and it is simultaneously one quarter of the unique CM value of the j -function at the discriminant $D = -4$ point $\tau = i$.

The present addendum makes this identification the foundation of a new structural reading of G_1 .

1.3 Overview of results

The main results, in logical order, are the following:

- **Theorem 2.1** (the $j(i)$ reformulation): $G_1 = J_{\text{short}}^3/(4\pi\alpha^{-1}) - 1 = j(i)/(4\Omega) - 1$.
- **Proposition 2.3** (breath period reconstruction): $\Omega = j(i)/(4(1 + G_1))$.
- **Proposition 3.1** (near-miss to $J_{\text{short}}/2$): $j(i) - 4\Omega = 5.9510\dots < 6 = J_{\text{short}}/2$, with a strictly positive deficit that is transcendental.
- **Proposition 4.1** (polynomial-geometric identity): $4\Omega = (4\pi)\alpha^{-1} = 16\pi^4 + 4\pi^3 + 4\pi^2$; the factor 4π is the surface area of the unit sphere S^2 .
- **Proposition 5.1** (modular residual): G_1 is unreachable by any CM value of j at discriminant $D = -4$.
- **Corollary 6.1** (transcendence): G_1 is transcendental.

2 The $j(i)$ reformulation of G_1

Theorem 2.1 ($j(i)$ reformulation of G_1). *With $\Omega = \pi\alpha^{-1}$ as in (1), $j(i) = 1728$, and $J_{\text{short}} = 12$:*

$$G_1 = \frac{j(i)}{4\Omega} - 1 = \frac{J_{\text{short}}^3}{4\pi\alpha^{-1}} - 1. \quad (4)$$

Equivalently:

$$\Omega = \frac{j(i)}{4(1 + G_1)}, \quad (5)$$

expressing the breath period as a G_1 -corrected quarter of $j(i)$.

Proof. By (2), $G_1 = (432 - \Omega)/\Omega$. Since $j(i) = 1728$ (P158) and $j(i)/4 = 432$, we substitute $432 = j(i)/4$:

$$G_1 = \frac{j(i)/4 - \Omega}{\Omega} = \frac{j(i)}{4\Omega} - 1.$$

The second form $J_{\text{short}}^3/(4\pi\alpha^{-1}) - 1$ follows from $j(i) = J_{\text{short}}^3$ and $\Omega = \pi\alpha^{-1}$. Rearranging $G_1 + 1 = j(i)/(4\Omega)$ gives (5). $\square$

Remark 2.2. Equation (4) is a reformulation, not an independent result: it follows from the definition of G_1 together with P158's identity $j(i) = 1728$. Its content is geometric rather than computational: the integer 432 that has appeared in the TOE since the definition of G_1 is now identified as the quarter- j value $j(i)/4$ at the CM point $\tau = i$. This locates G_1 precisely in the interface between the TOE's geometric period Ω and the modular arithmetic of $\tau = i$.

Proposition 2.3 (Breath period reconstruction). *The breath period Ω is uniquely determined by $j(i)$ and G_1 :*

$$\Omega = \frac{j(i)}{4(1 + G_1)} = \frac{1728}{4(1 + G_1)}. \quad (6)$$

Conversely, G_1 is uniquely determined by $j(i)$ and Ω . The pair $(j(i), G_1)$ encodes the same information as the pair $(432, \Omega)$.

Proof. Equation (6) is (5) restated. The converse is (2). The bijection between $(j(i), G_1)$ and $(432, \Omega)$ follows from $j(i)/4 = 432$ being rational. $\square$

3 The near-miss $j(i) - 4\Omega \approx J_{\text{short}}/2$

3.1 Numerical value and strict non-exactness

Proposition 3.1 (Near-miss to $J_{\text{short}}/2$). *The absolute gap satisfies*

$$j(i) - 4\Omega = 1728 - 4\pi\alpha^{-1} = 5.9510191304042890397\dots, \quad (7)$$

which is close to but strictly less than $J_{\text{short}}/2 = 6$. More precisely:

$$\frac{J_{\text{short}}}{2} - (j(i) - 4\Omega) = 6 - (1728 - 4\pi\alpha^{-1}) = 4\pi\alpha^{-1} - 1722 = 0.04898086959571096\dots \quad (8)$$

This deficit is strictly positive and transcendental.

Proof. Compute $j(i) - 4\Omega = 1728 - 4\pi(4\pi^3 + \pi^2 + \pi) = 1728 - 16\pi^4 - 4\pi^3 - 4\pi^2$. To 55 significant figures, using `verify_G1_modular.py` with `mp.dps = 60`:

$$j(i) - 4\Omega = 5.95101913040428903971145271280803465420267734452966\dots$$

The claim $j(i) - 4\Omega < J_{\text{short}}/2 = 6$ is equivalent to $4\pi\alpha^{-1} > 1722$. Numerically $4\pi\alpha^{-1} \approx 1722.049$, confirming the strict inequality.

For non-exactness: the claim $j(i) - 4\Omega = 6$ would require $4\pi\alpha^{-1} = 1722$, i.e., $\alpha^{-1} = 1722/(4\pi) \approx 137.028$. But the TOE constant is $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036$; these are unequal because $4\pi^3 + \pi^2 + \pi \neq 1722/(4\pi)$ is equivalent to $16\pi^4 + 4\pi^3 + 4\pi^2 \neq 1722$, which holds since the left side is transcendental and $1722 \in \mathbb{Z}$.

The deficit $4\pi\alpha^{-1} - 1722 = 16\pi^4 + 4\pi^3 + 4\pi^2 - 1722$ is transcendental because it is a nonconstant polynomial in π with rational coefficients minus a rational, and π is transcendental. $\square$

Remark 3.2. The near-miss $j(i) - 4\Omega \approx 6 = J_{\text{short}}/2$ is one more instance of the pattern of near-integer or near- J_{short} values that runs through the TOE addenda corpus. The deficit ≈ 0.04898 is approximately $\pi^2/200 \approx 0.04935$, though no closed form in obvious TOE quantities appears to be exact. This near-miss is purely observational; it is not a theorem.

3.2 Exact 55-digit value

For the record, to 55 significant figures (verified by `verify_G1_modular.py`):

$$\Omega = 430.5122452173989277400721368217979913364493306638675851\dots \quad (9)$$

$$G_1 = 0.003455778085591479075380623365743477970730119293270778\dots \quad (10)$$

$$j(i) - 4\Omega = 5.951019130404289039711452712808034654202677344529660\dots \quad (11)$$

4 The polynomial identity and S^2 geometry

Proposition 4.1 (Polynomial-geometric identity). *The following identities hold:*

$$4\Omega = 4\pi\alpha^{-1} = 16\pi^4 + 4\pi^3 + 4\pi^2, \quad (12)$$

and

$$4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}, \quad (13)$$

where $\text{vol}(S^2) = 4\pi$ is the surface area of the unit sphere in $\mathbb{R}^3$.

Proof. Substituting $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ into $4\pi\alpha^{-1}$:

$$4\pi(4\pi^3 + \pi^2 + \pi) = 16\pi^4 + 4\pi^3 + 4\pi^2.$$

The identity $\text{vol}(S^2) = 4\pi$ is the standard formula for the surface area of the unit 2-sphere. Hence $4\pi\alpha^{-1} = \text{vol}(S^2) \cdot \alpha^{-1}$. $\square$

Remark 4.2. The geometric reading of (13) is as follows. The TOE is formulated on S^3 , the 3-sphere. The quantity α^{-1} is a dimensionless coupling derived from S^3 geometry. The breath period $\Omega = \pi\alpha^{-1}$ has π as the ratio of the circumference to the diameter; $4\pi = \text{vol}(S^2)$ is the area of the equatorial 2-sphere inside S^3 . The identity $4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}$ therefore says that the 4Ω slot — the modular reference value $j(i)$ displaced by G_1 from 4Ω — is the product of the S^2 volume and the fine-structure denominator. Whether this has a direct interpretation in the S^3 Hopf fibration (with S^2 as the base and S^1 as the fibre) is left as an open direction.

5 G_1 as a modular residual

5.1 The CM point $\tau = i$ and its j -value

Recall the basic facts about the CM point $\tau = i$, established in P156–P159. The discriminant of $\mathbb{Q}(i)/\mathbb{Q}$ is $D = -4$. The j -function at this point takes the value

$$j(i) = 1728 \in \mathbb{Z}, \tag{14}$$

an integer — this is the defining property of CM points: j is an algebraic integer at every CM point. The value $j(i) = 1728$ is the unique value associated to the lattice $\mathbb{Z}[i]$ (the Gaussian integers), and it is the smallest CM value in absolute terms (for imaginary quadratic fields of small discriminant).

5.2 The modular residual interpretation

Proposition 5.1 (G_1 as modular residual at $\tau = i$). *The G_1 gap is the fractional deviation of the TOE breath period from the CM modular slot $j(i)/4 = 432$:*

$$G_1 = \frac{j(i)/4 - \Omega}{\Omega} = \frac{j(i)}{4\Omega} - 1. \tag{15}$$

No modular form f for $\text{SL}(2, \mathbb{Z})$ evaluates to 4Ω at any CM point τ_0 of discriminant $D = -4$ (i.e., at $\tau_0 = i$). Equivalently, 4Ω does not lie in the ring of special values $\{j(\tau_0)^{k/n} : k, n \in \mathbb{Z}_{>0}\}$ computed at $\tau_0 = i$.

Proof. The first statement is Theorem 2.1.

For the second statement: any CM value $f(i)$ of a modular form with algebraic Fourier coefficients is an algebraic number (by the CM theory of modular forms, see Shimura [1] or Silverman [2]). In particular, $j(i)^{k/n} = 1728^{k/n}$ is algebraic for all $k, n \in \mathbb{Z}_{>0}$. However, $4\Omega = 4\pi\alpha^{-1} = 16\pi^4 + 4\pi^3 + 4\pi^2$ is transcendental: it is a nonconstant polynomial in π with rational coefficients, and π is transcendental by Lindemann–Weierstrass. A transcendental number cannot equal an algebraic number. Hence 4Ω lies outside the field of CM values at $\tau = i$, and $G_1 = j(i)/(4\Omega) - 1$ is the residual measuring this gap. $\square$

Remark 5.2. This is the precise sense in which G_1 is a *modular residual*: it is not the value of any modular invariant at the relevant CM point, and it cannot be made zero by any algebraic adjustment to the lattice $\mathbb{Z}[i]$. The gap between Ω (a transcendental geometric period) and $j(i)/4$ (an integer CM value) is irreducible in the same sense as a period of an elliptic curve is transcendental while the j -invariant of that curve is algebraic. The present addendum establishes that G_1 occupies precisely this interface.

6 Transcendence of G_1

Corollary 6.1 (G_1 is transcendental). *G_1 is a transcendental number.*

Proof. By (4), $G_1 = j(i)/(4\Omega) - 1 = 1728/(4\Omega) - 1 = 432/\Omega - 1$. Since $432 \in \mathbb{Q}$ and Ω is transcendental (Lindemann–Weierstrass; $\Omega = \pi\alpha^{-1}$, a product of the transcendental π and the nonzero algebraic α^{-1} ... but wait: $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ is itself transcendental, being a nonconstant polynomial in π), Ω is transcendental. The ratio $432/\Omega$ of a nonzero rational by a transcendental is transcendental. Subtracting the integer 1 preserves transcendence. Hence $G_1 = 432/\Omega - 1$ is transcendental. $\square$

Remark 6.2. The transcendence of G_1 is not surprising given that Ω is transcendental, but it has a conceptual consequence: G_1 will never appear as a root of a polynomial with rational (or even algebraic) coefficients. In particular, it cannot be a rational number, cannot be $1/n$ for any integer n , and cannot be expressed in radicals or as a CM value. The TOE framework treats G_1 as a structural constant of the same transcendental type as π : it is not a fitting parameter but a definite mathematical number whose value is forced by the TOE geometry.

7 Connection to Ramanujan tau (P165)

Remark 7.1 (Ramanujan tau and 432). Addendum 165 (P165) observed that $\tau(3) = 252 = 7 \times 36 = 7 \times B_{F_4}$, where $\tau(n)$ denotes the Ramanujan tau function and $B_{F_4} = B_{F_4} = 36$ is the F_4 branching factor of P165. In the present context, $252 = 432 - 180 = 432 - 4 \times 45$, and separately

$$\frac{\tau(3)}{j(i)/4} = \frac{252}{432} = \frac{7}{12} = \frac{7}{J_{\text{short}}}. \quad (16)$$

The ratio $\tau(3)/432 = 7/12$ expresses the third Ramanujan coefficient as a rational multiple of J_{short}^{-1} of $432 = j(i)/4$. We record this as an observation: whether the factor 7 admits a root-system or moonshine interpretation complementary to $J_{\text{short}} = 12$ in this ratio is open. No proof is claimed here beyond the numerical verification.

8 Summary and open directions

8.1 Main results

The central identity of this addendum is:

$$\boxed{G_1 = \frac{j(i)}{4\Omega} - 1 = \frac{J_{\text{short}}^3}{4\pi\alpha^{-1}} - 1, \quad \Omega = \frac{j(i)}{4(1 + G_1)}}. \quad (17)$$

The supporting results are:

$$j(i) - 4\Omega = 5.951019\dots < 6 = \frac{J_{\text{short}}}{2} \quad (\text{near-miss, strictly non-exact}), \quad (18)$$

$$4\Omega = (4\pi)\alpha^{-1} = \text{vol}(S^2) \cdot \alpha^{-1} \quad (\text{sphere-volume identity}), \quad (19)$$

$$G_1 \notin \overline{\mathbb{Q}} \quad (\text{transcendental}). \quad (20)$$

Together these establish that G_1 is the irreducible modular residual at $\tau = i$: it measures the gap between the TOE's geometric period Ω and the CM modular slot $j(i)/4$, and this gap is transcendental and unreachable by any modular or algebraic construction.

8.2 Open directions

1. *The near-miss deficit* ≈ 0.04898 . The deficit $6 - (j(i) - 4\Omega) = 4\pi\alpha^{-1} - 1722 \approx 0.04898$ has no obvious closed form in TOE quantities. Whether it is a rational linear combination of powers of π with a combinatorial interpretation (analogous to the way Ω itself is $\pi\alpha^{-1}$ with α^{-1} a cubic in π) is an open question.
2. *S^3 Hopf fibration interpretation of $4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}$* . The S^3 geometry of the TOE carries a Hopf fibration with S^2 base and S^1 fibre. The identity $4\Omega = \text{vol}(S^2) \cdot \alpha^{-1}$ places 4Ω on the base sphere. Whether the Hopf lapse function $m = \sqrt{1 - u^2}$ (GON, Paper 30) integrates to 4Ω over the S^2 base in any natural sense is an open geometric question.
3. *Higher CM points*. The present addendum focuses on $\tau = i$ (discriminant $D = -4$). The CM points of other discriminants give different algebraic values of j . For example, $j(e^{2\pi i/3}) = 0$ (discriminant $D = -3$), and $j((1 + i\sqrt{7})/2) = -3375$ (discriminant $D = -7$). Whether the breath period Ω stands in a similarly structured near-miss relationship to quarter- j values at other CM points is not known.
4. *Tau-function connection*. Remark 7.1 records that $\tau(3)/432 = 7/J_{\text{short}}$. This ratio connects the Ramanujan tau function at prime 3 to the modular reference value $432 = j(i)/4$. A structural explanation — possibly via the McKay–Thompson series for the Monster group evaluated at $\tau = i$ (P163) — is an open direction.
5. *Period integrals*. In classical CM theory, the period of an elliptic curve E with CM by $\mathbb{Z}[i]$ is $\omega = \Gamma(1/4)^2/(4\sqrt{\pi})$, which is transcendental and unrelated to Ω by any known algebraic identity. Whether the ratio ω/Ω carries a number-theoretic meaning is open.

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