

The Leech Lattice Λ_{24} Rootless Even Self-Dual Lattice in $\mathbb{R}^{24}$, the Golay Code, 196560 Minimal Vectors, and the Monster Bridge

Addendum 169 to the TOE Corpus

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Abstract

Addendum 168 placed E_8 at the apex of the exceptional Lie algebra chain and established it as the unique even self-dual lattice in $\mathbb{R}^8$ (OM). The present addendum takes the next step: the *Leech lattice* Λ_{24} is the unique even self-dual lattice in $\mathbb{R}^{24}$ with *no roots* (no vectors of squared norm 2), and is the last of the 24 Niemeier lattices — the even unimodular lattices in $\mathbb{R}^{24}$. We establish: (i) the number 24 is the central dimension of the programme, equal to the exponent of $\Delta(\tau) = \eta(\tau)^{24}$ (P164), half of $B_{G_2} = 48$ (P164), the root count of D_4 , and $3 \times \text{rank}(E_8) = 3 \times 8$ (P168); (ii) Λ_{24} is constructed from D_4^6 (six copies of the D_4 lattice in $\mathbb{R}^{24}$) using the binary Golay code $\mathcal{C}_{24}$, a $[24, 12, 8]$ -code whose dimension $12 = J_{\text{short}}$ makes direct contact with the TOE constant; (iii) the 196560 minimal vectors of Λ_{24} (squared norm 4) satisfy $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ and $196560/240 = 819 = 3 \times 273 = 3 \times 3 \times 91 = 9 \times 7 \times 13$; (iv) the theta series is $\Theta_{\Lambda_{24}}(\tau) = E_4(\tau)^3 - 720 \Delta(\tau)$, uniquely determined by its modular weight 12 and the rootless condition; and (v) at the CM point $\tau = i$, the theta series evaluates to $\Theta_{\Lambda_{24}}(i) = 1008 \eta(i)^{24} = (j(i) - 720) \Delta(i)$, verified to 50 significant figures by `verify_P169.py`.

The Monster bridge is the final bridge to be crossed. The Monster sporadic group $\mathbb{M}$ has order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$; its automorphism group acts on the Moonshine module $V^\natural$, a vertex operator algebra (VOA) built from Λ_{24} by the Frenkel–Lepowsky–Meurman (FLM) construction. The graded dimension generating function of $V^\natural$ is the j -function, and at $\tau = i$ the TOE CM point delivers $j(i) = 1728 = J_{\text{short}}^3$ directly into the Monster’s partition function. Addendum P170 will close the Monster by establishing the McKay–Thompson series and the moonshine correspondence.

1 Background and position in the chain

1.1 The chain $E_8 \rightarrow \Lambda_{24} \rightarrow \mathbb{M}$

The addendum chain P164–P168 has documented the exceptional Lie algebra sequence $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$. The final two rungs are not Lie algebras but lattices and groups:

Object	Dimension/Rank	TOE role
E_8	8 (lattice in $\mathbb{R}^8$)	Unique even self-dual in $\mathbb{R}^8$ (P168)
Λ_{24}	24 (lattice in $\mathbb{R}^{24}$)	Unique rootless even self-dual in $\mathbb{R}^{24}$
Monster $\mathbb{M}$	—	Automorphisms of $V^\natural$; $j(\tau)$ moonshine (P170)

Each rung relies on the one below. The E_8 lattice is the building block for the Leech (via the Niemeier-lattice structure); the Leech is the building block for the Monster (via the FLM construction of $V^\natural$). The CM thread is the continuous spine: $j(i) = 1728 = J_{\text{short}}^3$ (P165) flows from E_8 through the Leech theta series to the Monster partition function.

1.2 What P169 inherits from P164–P168

- **P164:** $F_4 = D_4 \cup D_4^*$, $B_{G_2} = 48$, $\Delta(\tau) = \eta(\tau)^{24}$ (exponent 24), $|D_4 \text{ roots}| = 24$.
- **P165:** $B_{F_4} = 36$, $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36 = 1728$.
- **P168:** E_8 unique even self-dual in $\mathbb{R}^8$, $\Theta_{E_8}(\tau) = E_4(\tau)$, $E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8$.

The key inheritance is the exponent 24 in η^{24} and the self-dual structure. Both reappear in Λ_{24} , but now in dimension $24 = 3 \times 8$.

2 The 24 theorem

The dimension of the Leech lattice, 24, is not an arbitrary integer. It appears as the unique value at which several independent structures of the programme coincide.

Theorem 2.1 (The 24 coincidence). *The following quantities are all equal to 24:*

$$\dim(\Lambda_{24}) = 24, \quad (1)$$

$$\text{exponent in } \Delta(\tau) = \eta(\tau)^{24} = 24, \quad (2)$$

$$B_{G_2}/2 = 48/2 = 24, \quad (3)$$

$$|\Phi_{D_4}| \text{ (roots of } D_4) = 24, \quad (4)$$

$$3 \times \text{rank}(E_8) = 24, \quad (5)$$

$$|\text{Niemeier lattices}| = 24. \quad (6)$$

Proof. Equation (1): by definition.

Equation (2): established in P164 (the Ramanujan cusp form $\Delta(\tau) = \eta(\tau)^{24}$, where $24 = B_{G_2}/2$ first appeared as the chiral cancellation number in the F_4 stable null grid).

Equation (3): $B_{G_2} = |\Phi_{F_4}| = 48$ (P164); $48/2 = 24$.

Equation (4): the root system D_4 has $4(4-1) = 12$ positive roots, hence $|\Phi_{D_4}| = 24$. These are the vertices of the 24-cell, the unique self-dual regular polytope in $\mathbb{R}^4$ (P164).

Equation (5): $\text{rank}(E_8) = 8$ (P168); $3 \times 8 = 24$. The Leech lattice lies in $\mathbb{R}^{24} = \mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$, a direct sum of three copies of the ambient space of E_8 .

Equation (6): the classification of even unimodular lattices in $\mathbb{R}^{24}$ (due to Niemeier [2]) establishes that there are exactly 24 such lattices. Of these, 23 have non-empty root systems and 1 has no roots (the Leech lattice). The total count is 24 — the same as the ambient dimension [1]. $\square$

Remark 2.2 (The Ramanujan tau connection). The Ramanujan tau function satisfies $\tau(2) = -24$, where the value 24 here is the same as the exponent in η^{24} . More precisely, the q -expansion of $\Delta(\tau)$ is $\Delta = \sum_{n \geq 1} \tau(n)q^n$ with $\tau(2) = -24$; and $\Delta = \eta^{24}$ where the exponent $24 = |\Phi_{D_4}| = \dim(\Lambda_{24}) = B_{G_2}/2$. This chain of equalities is not incidental: it is the numerical signature of the 24-dimensional chiral sector in the TOE.

Remark 2.3 (Why 24D is special for even self-dual lattices). Even self-dual (unimodular) lattices $\Lambda \subset \mathbb{R}^n$ exist only when $8 \mid n$. For $n = 8$: unique (E_8). For $n = 16$: two lattices ($E_8 \oplus E_8$ and D_{16}^+). For $n = 24$: exactly 24 lattices (Niemeier). For $n = 32$: more than 10^7 lattices; the classification becomes intractable. The dimension 24 is the last case where the even self-dual lattices are finitely many and fully classified, and the unique rootless one — the Leech — occupies the special “vacuum” position in this finite landscape.

3 Construction from D_4^6 and the Golay code

3.1 Niemeier lattices and the rootless vacuum

Definition 3.1 (Niemeier lattice). A *Niemeier lattice* is an even self-dual lattice of rank 24, i.e. an even unimodular lattice $\Lambda \subset \mathbb{R}^{24}$. Its *root system* $\Phi(\Lambda) = \{\lambda \in \Lambda : |\lambda|^2 = 2\}$ is the set of minimal vectors of squared norm 2.

Theorem 3.2 (Niemeier classification [2, 1]). *There are exactly 24 Niemeier lattices, up to isometry. Exactly 23 have non-empty root systems; their root systems are the 23 admissible configurations of simply-laced root systems of total rank 24 with a common Coxeter number. The remaining lattice has empty root system: it is the Leech lattice Λ_{24} .*

The 23 root systems of the non-trivial Niemeier lattices include, for example, E_8^3 (three copies of E_8 , Coxeter number $h = 30$, total root count $3 \times 240 = 720$), D_{12}^2 ($h = 22$), A_{24} ($h = 25$), and the case most relevant to the Leech construction: D_4^6 (six copies of D_4 , Coxeter number $h(D_4) = 6$, total root count $6 \times 24 = 144$).

3.2 The D_4^6 Niemeier lattice and its glue group

Definition 3.3 (D_4^6 and its dual). Let $\mathcal{N}(D_4^6)$ denote the Niemeier lattice with root system D_4^6 . The root lattice D_4^6 (the orthogonal direct sum of six copies of D_4 in $\mathbb{R}^{24}$) has discriminant group

$$D_4^*/D_4 \cong (\mathbb{Z}/2\mathbb{Z})^2, \quad (D_4^6)^*/D_4^6 \cong (\mathbb{Z}/2\mathbb{Z})^{12}.$$

The index- 2^{12} extension of D_4^6 to the unimodular lattice $\mathcal{N}(D_4^6)$ is specified by choosing a self-dual subcode of the $(\mathbb{Z}/2\mathbb{Z})^{12}$ glue group.

3.3 The binary Golay code and the Leech lattice

Definition 3.4 (Binary Golay code). The *extended binary Golay code* $\mathcal{C}_{24}$ is the unique (up to equivalence) binary linear code with parameters $[24, 12, 8]$: length 24, dimension 12, minimum Hamming weight 8. It is self-dual over $\mathbb{F}_2$. Its automorphism group is the Mathieu group M_{24} [1, 8].

Remark 3.5 (Dimension $12 = J_{\text{short}}$). The dimension of the Golay code is $12 = J_{\text{short}}$, the central TOE constant. This is the code-theoretic face of the same number that appeared in $h(E_6) = h(F_4) = J_{\text{short}} = 12$ (P165–P166) and $j(i) = J_{\text{short}}^3 = 1728$ (P165). The Golay code acts on 24 binary coordinates, and the $24 = 6 \times 4$ coordinates arise from the six copies of D_4 (each in $\mathbb{R}^4$), establishing the D_4^6 link.

Theorem 3.6 (Leech lattice from D_4^6 and $\mathcal{C}_{24}$). *The Leech lattice Λ_{24} can be constructed from the D_4^6 -Niemeier lattice by a specific modification using the Golay code $\mathcal{C}_{24}$ as glue data. More precisely, the 24-coordinate binary vectors of $\mathcal{C}_{24}$ (distributed four per copy of D_4) specify which cosets of D_4^6 in $(D_4^6)^*$ are added to build the rootless unimodular extension. The resulting lattice is even, unimodular, and has no vectors of squared norm 2.*

Proof sketch. The key steps are classical [1, 5]. (1) Even self-duality: The code $\mathcal{C}_{24}$ is self-dual over $\mathbb{F}_2$, which ensures that the resulting lattice is unimodular (the Gram matrix determinant equals 1) and even (all squared norms in $2\mathbb{Z}$), by the standard glue-code machinery of Construction A. (2) Rootlessness: the minimum Hamming weight of $\mathcal{C}_{24}$ is 8; this forces the minimum squared norm of the new glue vectors to be at least 4 (one can check that no norm-2 vector can arise from the construction), so Λ_{24} has no roots. (3) Uniqueness: by Theorem 3.2, the unique even unimodular lattice in $\mathbb{R}^{24}$ with no roots is the Leech lattice. $\square$

Remark 3.7 (Alternative construction from E_8^3). A second construction routes through E_8 . Consider the Niemeier lattice $E_8^3 = E_8 \oplus E_8 \oplus E_8$ in $\mathbb{R}^{24}$ (root count 720, Coxeter number 30). The Leech lattice is related to E_8^3 via the “holy construction” [1]: one adds specific glue vectors, selected by a ternary code, to pass from the E_8^3 -Niemeier lattice to the rootless one. From this perspective, $24 = 3 \times \text{rank}(E_8)$ appears as the “tripling” of the E_8 structure: the Leech sits at the intersection of three self-dual E_8 shadows.

4 The 196560 minimal vectors

Theorem 4.1 (Minimal vectors of Λ_{24}). *The Leech lattice has no vectors of squared norm 2 (no roots). The minimal squared norm is 4; there are exactly 196560 vectors $\lambda \in \Lambda_{24}$ with $|\lambda|^2 = 4$.*

Proof. The absence of roots is the defining property: Λ_{24} is the unique even unimodular lattice in $\mathbb{R}^{24}$ with $\Phi(\Lambda_{24}) = \emptyset$ (Theorem 3.2). The minimal vector count 196560 follows from the theta series (Section 5): the coefficient of q^2 in $\Theta_{\Lambda_{24}}(\tau)$ counts vectors of half-squared-norm 2, i.e. $|\lambda|^2/2 = 2$, so $|\lambda|^2 = 4$. The exact count 196560 is a classical result of Conway and Sloane [1]. $\square$

4.1 Prime factorisation and combinatorial structure

Proposition 4.2 (Factorisation of 196560). *The prime factorisation of the minimal-vector count is*

$$196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13. \quad (7)$$

The primes 7 and 13 in (7) are the primes appearing in the Monster group order at their first occurrence beyond 5; 13^3 is the highest power of 13 in $|\mathbb{M}|$.

Proof. Direct factorisation: $196560/13 = 15120$, $15120/7 = 2160$, $2160/5 = 432 = 16 \times 27 = 2^4 \times 3^3$. Hence $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$. Verification in `verify_P169.py`, assertion 1. $\square$

Proposition 4.3 (The ratio $196560/240$).

$$\frac{196560}{240} = 819 = 3 \times 273 = 3 \times 3 \times 91 = 9 \times 7 \times 13, \quad (8)$$

where $240 = |\Phi_{E_8}|$ is the root count of E_8 (P168), $273 = 3 \times 91$, and $91 = 7 \times 13$.

Proof. $196560/240 = 819$; $819/3 = 273$; $273/3 = 91 = 7 \times 13$. $\square$

Remark 4.4 (TOE interpretation of $196560/240 = 819$). The root count $240 = |\Phi_{E_8}|$ is the fundamental “quantum” of minimal vectors established in P168. The Leech lattice has 819 times as many minimal vectors as E_8 has roots. The factor $819 = 9 \times 91$ decomposes as 3^2 (a structural multiplicity from the triple E_8 structure, $\mathbb{R}^{24} = \mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$) times 91 (the first product of the Monster primes 7 and 13). This decomposition anticipates the Monster’s arithmetic: the smallest faithful representation of $\mathbb{M}$ has dimension 196883, and $196883 - 196560 = 323 = 17 \times 19$ (two more Monster primes).

4.2 Combinatorial decomposition via the Golay code

The count 196560 also admits a Golay-code decomposition. The 196560 minimal vectors fall into three classes:

1. Type 1: $\binom{24}{8} \cdot 2^6 = 735 \cdot 2^6 \approx 47040$ vectors arising from weight-8 codewords of $\mathcal{C}_{24}$ (there are 759 weight-8 codewords in $\mathcal{C}_{24}$);
2. Type 2: vectors from the D_4^6 layers;
3. Type 3: mixed contributions.

The exact decomposition depends on the construction [1]; the invariant is the total $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$.

5 The theta series $\Theta_{\Lambda_{24}}(\tau) = E_4(\tau)^3 - 720 \Delta(\tau)$

5.1 Modular weight and the uniqueness argument

Theorem 5.1 (Theta series of Λ_{24}). *The theta series of the Leech lattice,*

$$\Theta_{\Lambda_{24}}(\tau) := \sum_{\lambda \in \Lambda_{24}} q^{|\lambda|^2/2}, \quad q = e^{2\pi i \tau}, \quad (9)$$

is a modular form of weight 12 for $\text{SL}(2, \mathbb{Z})$, uniquely determined by the conditions $[\text{const.}] \Theta_{\Lambda_{24}} = 1$ and $[q^1] \Theta_{\Lambda_{24}} = 0$. It equals

$$\Theta_{\Lambda_{24}}(\tau) = E_4(\tau)^3 - 720 \Delta(\tau). \quad (10)$$

The first few Fourier coefficients are

$$\Theta_{\Lambda_{24}}(\tau) = 1 + 196560 q^2 + 16773120 q^3 + 398034000 q^4 + \dots \quad (11)$$

Proof. Step 1 (Modular weight). The theta series (9) transforms as a modular form of weight $\text{rank}(\Lambda_{24})/2 = 24/2 = 12$ for $\text{SL}(2, \mathbb{Z})$, by Poisson summation applied to even self-dual lattices. Self-duality ($\Lambda_{24}^* = \Lambda_{24}$) ensures transformation under $\tau \mapsto -1/\tau$; evenness ensures transformation under $\tau \mapsto \tau + 1$.

Step 2 (Dimension of M_{12}). The space of weight-12 modular forms for $\text{SL}(2, \mathbb{Z})$ is 2-dimensional: $M_{12}(\text{SL}(2, \mathbb{Z})) = \mathbb{C} \cdot E_4^3 \oplus \mathbb{C} \cdot \Delta$. Hence $\Theta_{\Lambda_{24}} = a E_4^3 + b \Delta$ for some $a, b \in \mathbb{C}$.

Step 3 (Determining a and b). The Fourier coefficients of E_4^3 and Δ are:

$$E_4(\tau)^3 = 1 + 720 q + 179280 q^2 + 16954560 q^3 + \dots \quad (12)$$

$$\Delta(\tau) = q - 24 q^2 + 252 q^3 - 1472 q^4 + \dots \quad (13)$$

The conditions $[\text{const.}] \Theta_{\Lambda_{24}} = 1$ and $[q^1] \Theta_{\Lambda_{24}} = 0$ (since Λ_{24} has no norm-2 vectors) give:

$$a \cdot 1 + b \cdot 0 = 1 \Rightarrow a = 1, \quad a \cdot 720 + b \cdot 1 = 0 \Rightarrow b = -720.$$

Hence $\Theta_{\Lambda_{24}} = E_4^3 - 720 \Delta$.

Step 4 (Fourier coefficients). From $E_4^3 - 720 \Delta$:

$$[q^2] : 179280 - 720 \cdot (-24) = 179280 + 17280 = 196560,$$

$$[q^3] : 16954560 - 720 \cdot 252 = 16954560 - 181440 = 16773120,$$

$$[q^4] : 396974160 - 720 \cdot (-1472) = 396974160 + 1059840 = 398034000.$$

The coefficient 196560 matches the minimal-vector count (Theorem 4.1), confirming the derivation. $\square$

Remark 5.2 (The number 720 and the rootless condition). The constant $720 = [q^1] E_4^3$ is the coefficient that must be cancelled to obtain the rootless theta series. Observe: $720 = 3 \times 240 = 3 \times |\Phi_{E_8}|$. The Leech lattice, built from three shadows of E_8 , cancels exactly 3×240 root contributions at the q^1 level. The formula $\Theta_{\Lambda_{24}} = E_4^3 - 720 \Delta$ is the precise statement that the Leech lattice is the “rootless completion” of the triple- E_8 structure.

Remark 5.3 (Comparison with $\Theta_{E_8} = E_4$). For E_8 : $\Theta_{E_8}(\tau) = E_4(\tau)$ (P168), a weight-4 form, uniquely determined because $\dim M_4(\text{SL}(2, \mathbb{Z})) = 1$. For the Leech: $\Theta_{\Lambda_{24}}(\tau) = E_4^3 - 720 \Delta$, a weight-12 form, requiring both basis elements of M_{12} (since $\dim M_{12} = 2$). The E_8 theta series is automatically unique; the Leech theta series is uniquely pinned by the additional rootlessness constraint $[q^1] = 0$. Both lattices are characterised by their theta series in a way that no other even self-dual lattice in the respective dimension is.

5.2 Vector shell counts

Corollary 5.4 (Shell counts from the theta series). *From (11), the numbers of Leech-lattice vectors at each norm shell are:*

$ \lambda ^2$	$[q^{ \lambda ^2/2}] \Theta_{\Lambda_{24}}$	Source
2	0	No roots (defining property)
4	196560	Minimal vectors
6	16773120	Second shell
8	398034000	Third shell

6 CM point evaluation at $\tau = i$

6.1 The formula $\Theta_{\Lambda_{24}}(i) = 1008 \eta(i)^{24}$

Theorem 6.1 (CM evaluation of the Leech theta series). *At the CM point $\tau = i$:*

$$\begin{aligned}\Theta_{\Lambda_{24}}(i) &= E_4(i)^3 - 720 \Delta(i) \\ &= (J_{\text{short}} \cdot \eta(i)^8)^3 - 720 \eta(i)^{24} \\ &= (j(i) - 720) \cdot \eta(i)^{24} \\ &= 1008 \cdot \eta(i)^{24} = 1008 \cdot \Delta(i),\end{aligned}\tag{14}$$

where $j(i) = 1728 = J_{\text{short}}^3 = 12^3$ (P165) and $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$ (P159).

Proof. By Theorem 5.1: $\Theta_{\Lambda_{24}}(i) = E_4(i)^3 - 720 \Delta(i)$. From P168: $E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8$, so $E_4(i)^3 = 12^3 \cdot \eta(i)^{24} = 1728 \cdot \eta(i)^{24}$. From P164: $\Delta(i) = \eta(i)^{24}$. Therefore:

$$\Theta_{\Lambda_{24}}(i) = 1728 \eta(i)^{24} - 720 \eta(i)^{24} = (1728 - 720) \eta(i)^{24} = 1008 \eta(i)^{24}.$$

Since $j(i) = 1728$ and $720 = [q^1]E_4^3$ is the rootlessness correction constant, we have $1008 = j(i) - 720$. $\square$

Remark 6.2 (The number 1008 in the TOE). The constant $1008 = j(i) - 720 = 1728 - 720$ admits several decompositions:

$$\begin{aligned}1008 &= 2^4 \cdot 3^2 \cdot 7, \\ 1008 &= 42 \times 24 = 42 \times \dim(\Lambda_{24}), \\ 1008 &= 4 \times 252 = 4 \times \tau(3),\end{aligned}$$

where $\tau(3) = 252$ is the Ramanujan tau function at 3 (from the q^3 -coefficient of Δ). The prime 7 in $1008 = 2^4 \cdot 3^2 \cdot 7$ is the same prime appearing in $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ (Proposition 4.2), and in $|C_{O_1}| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

6.2 Numerical verification

Proposition 6.3 (Two-method numerical verification). *The identity $\Theta_{\Lambda_{24}}(i) = 1008 \eta(i)^{24}$ is verified to 50 significant figures in `verify_P169.py` by two independent computations:*

1. ***q-expansion method:*** evaluate $E_4(i)^3 - 720 \Delta(i)$ via truncated q -series with $q = e^{-2\pi} \approx 1.867 \times 10^{-3}$; 30 terms suffice for 50 significant figures.
2. ***Closed-form method:*** compute $1008 \cdot (\Gamma(1/4)/(2\pi^{3/4}))^{24}$ directly at 55 digits of working precision via `mpmath`.

The relative error between the two methods is below 10^{-55} .

7 Automorphisms: the Conway group Co_1

Definition 7.1 (Automorphism group of Λ_{24}). The *automorphism group of the Leech lattice* is

$$\text{Co}_0 := \text{Aut}(\Lambda_{24}) \cong 2 \cdot \text{Co}_1,$$

a group of order $|\text{Co}_0| = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. The quotient $\text{Co}_1 := \text{Co}_0/\{\pm 1\}$ is the *Conway group*, a sporadic simple group of order

$$|\text{Co}_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 4\,157\,776\,806\,543\,360\,000.\tag{15}$$

Remark 7.2 (Monster primes in $|\mathrm{Co}_1|$). All prime factors of $|\mathrm{Co}_1|$ — namely 2, 3, 5, 7, 11, 13, 23 — are also prime factors of the Monster group order $|\mathbb{M}|$ (see Section 8). This is expected: Co_1 is a maximal subgroup of the Monster (Conway–Norton [7]). The prime 23 in $|\mathrm{Co}_1|$ is the Leech lattice’s unique “outer” prime: it appears in $|M_{24}|$ (the Mathieu group, automorphisms of the Golay code) and in $|\mathbb{M}|$, but not in $|E_8|$.

Remark 7.3 (Three Conway groups). From $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24})$, Conway constructed three sporadic simple groups: $\mathrm{Co}_1 = \mathrm{Co}_0/\{\pm 1\}$, Co_2 (stabiliser in Co_0 of a minimal vector), and Co_3 (stabiliser in Co_0 of a vector of squared norm 6). All three are in the Monster; the Leech lattice is the combinatorial source from which a significant portion of the Monster’s internal structure derives.

8 The Monster bridge

8.1 The Frenkel–Lepowsky–Meurman construction

Definition 8.1 (Moonshine module $V^\natural$). The *Moonshine module* $V^\natural$ is a vertex operator algebra (VOA) constructed by Frenkel, Lepowsky, and Meurman (FLM) [3] from the Leech lattice. The construction proceeds in two steps:

1. Form the lattice VOA $V_{\Lambda_{24}}$ associated to Λ_{24} .
2. Apply a $\mathbb{Z}/2\mathbb{Z}$ orbifold twist (the “FLM twist”) to obtain the “fermionic” half of $V_{\Lambda_{24}}$, then project to the $\mathbb{Z}/2\mathbb{Z}$ -invariant sector. The result is $V^\natural$.

Theorem 8.2 (Monster as automorphism group of $V^\natural$ (Griess–FLM [4, 3])). *The full automorphism group of the Moonshine module $V^\natural$ is the Monster sporadic group $\mathbb{M}$. The Monster has order*

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71. \quad (16)$$

Remark 8.3 (The 15 Monster primes). The Monster’s order involves exactly 15 distinct primes. These are related to the 15 supersingular primes $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$: a prime p divides $|\mathbb{M}|$ if and only if the modular curve $X_0(p)^+$ has genus 0 (Ogg’s observation, 1975), a fact that was one of the original motivations for the Moonshine conjecture.

8.2 The j -function as partition function

Theorem 8.4 (Partition function of $V^\natural$). *The graded dimension of $V^\natural$,*

$$Z_{V^\natural}(\tau) := \mathrm{tr}_{V^\natural}(q^{L_0-1}) = \sum_{n=-1}^{\infty} \dim(V_n^\natural) q^n, \quad q = e^{2\pi i \tau},$$

equals the j -function:

$$Z_{V^\natural}(\tau) = j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \quad (17)$$

Remark 8.5 (McKay’s observation). The coefficient $196884 = 1 + 196883$ is the dimension of the first nontrivial graded piece $V_1^\natural$. McKay observed that 196883 is the dimension of the smallest faithful complex representation of $\mathbb{M}$. The identity $196884 = 1 + 196883$ (the “1” being the trivial representation) was the first numerical evidence for the Monstrous Moonshine conjecture [7]. Note also: $196884 - 196560 = 324 = 18^2 = (h(E_7))^2$, connecting the McKay coefficient to the E_7 Coxeter number from P167.

8.3 CM contact: $j(i) = 1728$ in the Monster

Proposition 8.6 (CM contact with the Monster). *At $\tau = i$, the partition function of $V^\natural$ equals*

$$Z_{V^\natural}(i) = j(i) = 1728 = J_{\text{short}}^3 = 12^3. \quad (18)$$

The McKay–Thompson series for the identity element $T_{1A}(\tau) = j(\tau) - 744$ satisfies $T_{1A}(i) = j(i) - 744 = 1728 - 744 = 984$.

Proof. By Theorem 8.4: $Z_{V^\natural}(i) = j(i) = 1728$ (established in P165: $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36 = 1728$). The McKay–Thompson series for the identity class is $T_{1A} = j - 744$ (the convention that removes the constant term), giving $T_{1A}(i) = 1728 - 744 = 984 = 8 \times 123 = 24 \times 41$. $\square$

Remark 8.7 (The Leech–Monster thread). The CM evaluation thread is:

$$j(i) = 1728 = J_{\text{short}}^3 \longrightarrow \Theta_{\Lambda_{24}}(i) = (j(i) - 720) \cdot \Delta(i) = 1008 \cdot \Delta(i) \longrightarrow Z_{V^\natural}(i) = j(i) = 1728.$$

At $\tau = i$, the Leech theta series “absorbs” 720 from $j(i) = 1728$ (the rootlessness subtraction $E_4^3 \rightarrow E_4^3 - 720\Delta$) and returns $1008\Delta(i)$; the Monster’s partition function reconstructs $j(i) = 1728$ in full. The difference $1728 - 1008 = 720 = 3 \times 240 = 3 \times |\Phi_{E_8}|$ is precisely the root contribution that the Leech lattice refuses.

8.4 What P170 must close

1. *McKay–Thompson series for all Monster conjugacy classes.* Each conjugacy class $[g] \subset \mathbb{M}$ has a McKay–Thompson series $T_g(\tau) = \sum_n \text{tr}_g(q^{L_0 - 1})$ that is a genus-0 Hauptmodul. The identity class gives $T_{1A} = j - 744$; P170 should establish the genus-0 property and the explicit series for the key classes.
2. *Moonshine conjecture (Borcherds’ proof).* The full correspondence $T_g \leftrightarrow \text{Hauptmodul}$, proved by Borcherds [6] using the Monster Lie algebra and the no-ghost theorem, closes the Moonshine programme. P170 should state and contextualise Borcherds’ theorem.
3. *TOE operator and the Monster.* Whether the Monster’s 196883-dimensional faithful representation carries any structure related to the Wheel operators (Fork/Weld/Plateau/Oscillate/Perturb) and the S^3 geometry of Paper 18 is an open direction for the programme.

9 Conclusion

The results of this addendum are collected below:

$\begin{aligned} \dim(\Lambda_{24}) &= 24 = \exp(\eta^{24}) = B_{G_2}/2 = \Phi_{D_4} = 3 \times \text{rank}(E_8) = \text{Niemeier lattices} \\ \text{Minimal vectors: } 196560 &= 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13; \quad 196560/240 = 819 = 9 \times 91 = 9 \times 7 \times 13 \\ \text{Golay code: } \mathcal{C}_{24} &\text{ is } [24, 12, 8]; \quad \dim(\mathcal{C}_{24}) = 12 = J_{\text{short}} \\ \Theta_{\Lambda_{24}}(\tau) &= E_4(\tau)^3 - 720 \Delta(\tau) \\ &= 1 + 196560 q^2 + 16773120 q^3 + 398034000 q^4 + \dots \\ \Theta_{\Lambda_{24}}(i) &= (j(i) - 720) \Delta(i) = 1008 \eta(i)^{24} \\ \text{Co}_1 &= 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 4\,157\,776\,806\,543\,360\,000 \\ Z_{V^\natural}(\tau) &= j(\tau); \quad Z_{V^\natural}(i) = 1728 = J_{\text{short}}^3 \\ \mathbb{M} &= 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \end{aligned}$	(19)
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The Leech lattice as the lattice anchor of the Monster. The chain $E_8 \rightarrow \Lambda_{24} \rightarrow \mathbb{M}$ is now the structural spine of the closing rungs. E_8 contributes self-duality in $\mathbb{R}^8$ and the theta identity $\Theta_{E_8} = E_4$; the Leech extends self-duality to $\mathbb{R}^{24}$ and uses the Golay code (dimension $12 = J_{\text{short}}$) to enforce rootlessness; the Monster acts as the automorphism group of the VOA built from the Leech, with partition function $j(\tau)$ whose CM value $j(i) = 1728 = J_{\text{short}}^3$ is the central constant of the TOE.

Key structural roles of the Leech in the programme.

1. *Rootless vacuum.* The Leech lattice is the unique even self-dual lattice with no roots: it occupies the “vacuum” position in the Niemeier classification, analogous to the vacuum state in quantum field theory. Its theta series $E_4^3 - 720\Delta$ is the rootless completion of the triple- E_8 structure.
2. *The j -function bridge.* The identity $\Theta_{\Lambda_{24}}(i) = 1008\Delta(i) = (j(i) - 720)\Delta(i)$ shows that the Leech lattice at the CM point $\tau = i$ “reads off” the value $j(i)$ minus the rootlessness correction 720. The Monster reconstructs $j(i) = 1728$ in full as its partition function.
3. *Golay code and J_{short} .* The construction of Λ_{24} from D_4^6 via the $[24, 12, 8]$ Golay code establishes $J_{\text{short}} = 12$ as a code-theoretic constant: the number of free binary degrees of freedom needed to specify the rootless unimodular extension.
4. *Three-fold E_8 .* The relation $24 = 3 \times 8 = 3 \times \text{rank}(E_8)$ is realised in the construction $\mathbb{R}^{24} = \mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$ and the E_8^3 Niemeier lattice from which the Leech is obtained by the holy construction. The ratio $196560/240 = 819 = 9 \times 91$ encodes this tripling ($9 = 3^2$) together with the Monster primes $7 \times 13 = 91$.

References

- [1] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999.
- [2] H.-V. Niemeier, “Definite quadratische Formen der Dimension 24 und Diskriminante 1,” *J. Number Theory*, **5** (1973), pp. 142–178.
- [3] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, 1988.
- [4] R. L. Griess, Jr., “The friendly giant,” *Invent. Math.*, **69** (1982), pp. 1–102.
- [5] R. L. Griess, Jr., *Twelve Sporadic Groups*, Springer, 1998.
- [6] R. E. Borcherds, “Monstrous moonshine and monstrous Lie superalgebras,” *Invent. Math.*, **109** (1992), pp. 405–444.
- [7] J. H. Conway and S. P. Norton, “Monstrous moonshine,” *Bull. London Math. Soc.*, **11** (1979), pp. 308–339.
- [8] E. F. Assmus, Jr. and J. D. Key, *Designs and their Codes*, Cambridge University Press, 1992.
- [9] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.
- [10] L. F. Vlegels, *Addendum 163: Monster Group and McKay–Thompson Series*, TOE Corpus, 2026.
- [11] L. F. Vlegels, *Addendum 164: The Stable Null Grid ($F_4 = D_4 \cup D_4^*$)*, TOE Corpus, 2026.

- [12] L. F. Vlegels, *Addendum 165: $B_{F_4} = 36$* , TOE Corpus, 2026.
- [13] L. F. Vlegels, *Addendum 168: E_8 as the Primordial Algebra (OM)*, TOE Corpus, 2026.