

E_8 : The Primordial Algebra (OM) Root System, Self-Duality, and the Theta Series Identity $\Theta_{E_8}(\tau) = E_4(\tau)$

Addendum 168 to the TOE Corpus

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Abstract

The exceptional chain $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$ reaches its apex with E_8 . The preceding addenda established F_4 as the stable null grid (P164), $B_{F_4} = 36$ (P165), E_6 as the Jordan structure group with Coxeter number $h(E_6) = J_{\text{short}} = 12$ (P166), and E_7 as the Fano bridge with $h(E_7) = 18$ and the 56-dimensional Freudenthal triple system (P167). The present addendum places E_8 at the summit, justifying its designation as *OM*: the self-contained, self-dual primordial algebra.

We establish: (i) the exponents of E_8 are 1, 7, 11, 13, 17, 19, 23, 29, giving Coxeter number $h(E_8) = h^\vee(E_8) = 30$ and root count $|\Phi_{E_8}| = 240 = 8 \times 30 = \text{rank}(E_8) \times h(E_8)$; the canonical Coxeter-plane projection exhibits 30-fold dihedral symmetry with exactly 8 orbits; (ii) E_8 is the unique even self-dual lattice in $\mathbb{R}^8$, a property we call *primordial*: the lattice equals its own dual, $E_8^* = E_8$, with Gram-matrix determinant equal to 1; (iii) the theta series of the E_8 lattice equals the normalised weight-4 Eisenstein series, $\Theta_{E_8}(\tau) = E_4(\tau)$; at the CM point $\tau = i$ this gives $\Theta_{E_8}(i) = E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8$ (from P159), verified to 50 significant figures by `verify_P168.py`; (iv) the dimension $248 = 8 + 240 = \text{rank} + |\Phi_{E_8}|$ appears as the q^1 coefficient of $j(\tau)^{1/3}$ (from P161), making E_8 the first word that the j -function speaks; and (v) the symmetry-breaking cascade $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow F_4 \rightarrow G_2$ traces root sub-systems of sizes $240 \supset 126 \supset 72 \supset 48 \supset 12$. The Coxeter-number relation $h(E_8) = 30 = h(E_6) + h(E_7) = 12 + 18$ is proved and contextualised.

1 Background and position in the chain

1.1 The chain $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$

The exceptional Lie algebras form a unique finite sequence, and E_8 is its terminus. The table below collects the structural invariants built up across Addenda P151–P167.

Algebra	Rank	$ \Phi $	h	$h^\vee$	TOE role
G_2	2	12	6	4	Short-root sector, Fano symmetry
F_4	4	48	12	9	Stable null grid, $B_{G_2} = 48$ (P164)
E_6	6	72	12	12	Jordan structure group, $27^3 = j(i)$ (P166)
E_7	7	126	18	18	Fano bridge, Freudenthal triple system (P167)
E_8	8	240	30	30	Primordial (OM), self-dual (P168)

The Coxeter numbers of the chain are 6, 12, 12, 18, 30 (suppressing F_4 in the arithmetic sequence since $h(F_4) = 12 = h(E_6)$ despite F_4 not being simply laced). Three patterns stand out. First, $30 = 12 + 18 = h(E_6) + h(E_7)$ — the apex Coxeter number is the sum of the two preceding ones. Second, all Coxeter numbers are multiples of $6 = h(G_2)/1$, and $30 = 5 \times 6$. Third, the rank sequence 2, 4, 6, 7, 8 terminates with $8 = \text{rank}(E_8)$, which — unlike the Fano interruption at 7 — completes the octonion dimension count: $8 = \dim_{\mathbb{R}}(\mathbb{O})$.

1.2 Open threads closed by P168

Three threads were left open at the end of P167. First, P167 noted the decomposition $\mathbf{248} = \mathbf{133} + \mathbf{56}_{1/2} + \mathbf{56}_{-1/2} + \mathbf{1} + \mathbf{1}$ under $E_7 \times U(1)$; this paper establishes $|\Phi_{E_8}| = 240$ and $\dim(E_8) = 248$ from first principles (Section 2). Second, P164 introduced $\Delta(i) = \eta(i)^{24}$ and P159 established $E_4(i) = J_{\text{short}} \cdot \eta(i)^8$; the present paper connects both to E_8 via the identity $\Theta_{E_8}(\tau) = E_4(\tau)$ (Section 4). Third, P161 recorded 248 as the q^1 coefficient of $j^{1/3}$ without a derivation from the exceptional algebra chain; Section 6 supplies that context.

2 Root system and Coxeter structure of E_8

2.1 Exponents and root count

Theorem 2.1 (Coxeter data of E_8). *The exponents of E_8 are 1, 7, 11, 13, 17, 19, 23, 29. From these:*

$$h(E_8) = \max\{1, 7, 11, 13, 17, 19, 23, 29\} + 1 = 29 + 1 = 30, \quad (1)$$

$$|\Phi_{E_8}| = 2 \sum_j m_j = 2(1 + 7 + 11 + 13 + 17 + 19 + 23 + 29) = 2 \times 120 = 240. \quad (2)$$

Since E_8 is simply laced (all roots of equal length), $h^\vee(E_8) = h(E_8) = 30$.

Proof. For a simple Lie algebra of rank r , the exponents $m_1 \leq \dots \leq m_r$ satisfy: (a) $h = m_r + 1$ (where m_r is the largest exponent); (b) $|\Phi^+| = \sum_{j=1}^r m_j$ (positive root count equals sum of exponents); see [1, 2]. For E_8 , the exponents are classical (cf. the degrees of the basic polynomial invariants of E_8 , which are 2, 8, 12, 14, 18, 20, 24, 30, each equal to $m_j + 1$); they can be read from the extended E_8 Dynkin diagram or from the eigenvalues of the Coxeter element. Explicitly:

$$\{m_j\} = \{1, 7, 11, 13, 17, 19, 23, 29\}, \quad \sum m_j = 120 = |\Phi_{E_8}^+|, \quad m_8 = 29 \Rightarrow h = 30.$$

Since E_8 lies in the ADE series (specifically the E -series with simply-laced Dynkin diagram), every root has the same length as its coroot, so $h^\vee = h = 30$. $\square$

Remark 2.2 (The exponent 29). The largest exponent of E_8 is 29, which is prime. In the chain: G_2 has $m_r = 5$ (prime), F_4 has $m_r = 11$ (prime), E_6 has $m_r = 11$ (prime), E_7 has $m_r = 17$ (prime), E_8 has $m_r = 29$ (prime). All largest exponents in the exceptional chain are prime; this reflects the structure of the Weyl group action on the root system and has been observed but not given a simple uniform explanation.

2.2 The 8×30 factorisation

Theorem 2.3 (The primordial factorisation). *The root count and dimension of E_8 satisfy:*

$$|\Phi_{E_8}| = 240 = 8 \times 30 = \text{rank}(E_8) \times h(E_8), \quad (3)$$

$$\dim(E_8) = 248 = 8 + 240 = \text{rank}(E_8) + |\Phi_{E_8}|, \quad (4)$$

$$h(E_8) = 30 = 5 \times 6 = 5 \times h(G_2) = h(E_6) + h(E_7) = 12 + 18. \quad (5)$$

Proof. Equation (3): $|\Phi_{E_8}| = 240$ and $\text{rank}(E_8) = 8$, $h(E_8) = 30$ by Theorem 2.1; the identity $|\Phi| = \text{rank} \times h$ holds for all simply-laced algebras (positive roots arrange into r Coxeter orbits of size $h/2$, giving $|\Phi^+| = rh/2$, hence $|\Phi| = rh$; for E_8 : $240 = 8 \times 30$ with $|\Phi^+| = 120 = 8 \times 15 = 8 \times (30/2)$, confirmed).

Equation (4): $\dim(\mathfrak{g}) = r + |\Phi|$ for any simple Lie algebra (r -dimensional Cartan subalgebra plus one root space per root). For E_8 : $\dim(E_8) = 8 + 240 = 248$.

Equation (5): $30 = 5 \times 6$ is elementary. The additive relation $h(E_8) = h(E_6) + h(E_7)$ follows from Theorem 2.1 and the values $h(E_6) = 12$, $h(E_7) = 18$ (P166, P167). This additive relation reflects the height filtration of E_8 relative to the $E_6 \oplus E_7$ sub-root-system: the largest exponent of E_8 is $29 = 11 + 17 - 1 + 2$, related to the sum of the largest exponents of E_6 (11) and E_7 (17), though the exact decomposition of exponents under the $E_6 \oplus E_7 \subset E_8$ embedding is more intricate. The cleanest statement is at the Coxeter-number level: $30 = 12 + 18$. $\square$

Corollary 2.4 (The complete Coxeter chain). *The Coxeter numbers across the exceptional chain are:*

$$h(G_2) = 6, \quad h(F_4) = 12, \quad h(E_6) = 12, \quad h(E_7) = 18, \quad h(E_8) = 30.$$

All are multiples of 6. The subsequence 6, 12, 18, 30 (deleting the repeated 12) is $6 \times \{1, 2, 3, 5\}$, with differences 6, 6, 12 and final sum $h(E_6) + h(E_7) = h(E_8)$.

2.3 The 30-fold Coxeter plane projection

Theorem 2.5 (30-fold projection and 8 orbits). *The canonical Coxeter-plane projection of Φ_{E_8} exhibits 30-fold dihedral symmetry $\text{Dih}_{30} = I_2(30)$. The 240 roots arrange into exactly 8 orbits of 30 under the cyclic group $\langle c_{E_8} \rangle$ of order $h = 30$ generated by the Coxeter element.*

Proof. The Coxeter plane is the real eigenplane of the Coxeter element $c = s_1 \cdots s_8$ for the eigenvalues $e^{\pm 2\pi i/h} = e^{\pm 2\pi i/30}$, giving a rotation by $2\pi/30 = 12^\circ$ on this plane, hence 30-fold dihedral symmetry. For a simply-laced algebra, all roots project to the same magnitude in the Coxeter plane, and the orbit of any root under $\langle c \rangle$ has size $h = 30$. The number of orbits is $|\Phi_{E_8}|/h = 240/30 = 8 = \text{rank}(E_8)$. Thus the orbit count equals the rank, as for all simply-laced algebras. $\square$

Remark 2.6 (30-fold vs. 8-fold symmetry). The ambient space of E_8 is $\mathbb{R}^8$, which has the 8-fold symmetry of a hypercube. The Coxeter-plane projection, however, sees 30-fold symmetry — not 8-fold. This is a hallmark of E_8 : its geometry transcends the symmetry of its ambient dimension. The projection shows 8 concentric “rings” of 30 points each, producing the famous two-dimensional picture of the E_8 root system where all 240 roots appear as points at uniform distance from the origin, arranged in a regular 30-gon pattern.

3 Self-duality: E_8 as the primordial lattice

3.1 Even self-dual lattices in $\mathbb{R}^8$

Definition 3.1 (Even self-dual lattice). A lattice $\Lambda \subset \mathbb{R}^n$ is:

- *even*: if $|\lambda|^2 \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$ (all vectors have even squared norm);
- *self-dual* (or *unimodular*): if $\Lambda^* = \Lambda$, where the *dual lattice* is $\Lambda^* = \{x \in \mathbb{R}^n : \langle x, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$.

Equivalently, Λ is self-dual if and only if the Gram matrix $G_{ij} = \langle \lambda_i, \lambda_j \rangle$ of any basis $\{\lambda_1, \dots, \lambda_n\}$ has determinant $\det(G) = \pm 1$.

Theorem 3.2 (Uniqueness of E_8). *Up to isometry, E_8 is the unique even self-dual lattice in $\mathbb{R}^8$. In particular:*

1. $E_8^* = E_8$ (the lattice equals its own dual),
2. $\det(G_{E_8}) = 1$ (the Gram matrix of any basis has determinant 1),
3. E_8 is the densest lattice packing in $\mathbb{R}^8$, with packing density $\pi^4/384$.

Proof. The classification of even self-dual lattices in $\mathbb{R}^n$ is possible only when $8 \mid n$ (this is a number-theoretic constraint: the signature must be divisible by 8, which forces $n \equiv 0 \pmod{8}$). For $n = 8$, Milnor’s theorem [6] and the subsequent classification [4] establish that the unique even self-dual lattice in $\mathbb{R}^8$ is E_8 .

Self-duality $E_8^* = E_8$ follows from the fact that the E_8 root lattice is generated by the minimal vectors of the lattice (the roots, which all have squared norm 2) and is closed under the relevant inner products. The Cartan matrix of E_8 has determinant $\det(A_{E_8}) = 1$, which equals the squared volume of the fundamental domain; since the Gram matrix of the simple roots coincides with the Cartan matrix for a simply-laced algebra with all roots of length $\sqrt{2}$, we have $\det(G_{E_8}) = 1$. The packing result is classical (see [4, 9]). $\square$

Remark 3.3 (Why “primordial”). Self-duality $\Lambda^* = \Lambda$ means the lattice is *its own reciprocal*: it contains all the information needed to describe itself. In the TOE programme, we call E_8 *primordial* (OM) because this property is the lattice analogue of a self-referential fixed point: unlike G_2, F_4, E_6, E_7 , whose root lattices have non-trivial duals, E_8 needs no external data to complete itself. The OM designation also reflects the fact that E_8 contains all the others as sub-root-systems.

3.2 The Cartan matrix and its determinant

Definition 3.4 (Cartan matrix of E_8). Label the nodes of the E_8 Dynkin diagram $1, \dots, 8$ with Bourbaki [2] conventions, where the main chain runs 1–3–4–5–6–7–8 with a branch node 2 attached to node 4. The Cartan matrix A is the 8×8 integer matrix with $A_{ii} = 2$, $A_{ij} = -1$ if nodes i and j are adjacent in the Dynkin diagram, and $A_{ij} = 0$ otherwise. Explicitly:

$$A_{E_8} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Proposition 3.5 ($\det(A_{E_8}) = 1$). *The determinant of the E_8 Cartan matrix is $\det(A_{E_8}) = 1$.*

Proof. For any simple Lie algebra $\mathfrak{g}$ of rank r , the determinant of the Cartan matrix equals the order of the centre of the simply-connected Lie group G with Lie algebra $\mathfrak{g}$: $\det(A) = |Z(G)|$. The simply-connected form of E_8 has trivial centre, $Z(E_8) = \{1\}$, so $\det(A_{E_8}) = 1$ (see [3]). Equivalently, this is the index of the root lattice in the weight lattice, which for E_8 is 1 (the weight lattice equals the root lattice, since all fundamental weights are in the root lattice); this is a consequence of self-duality. The result can also be verified by direct computation of the 8×8 integer determinant (carried out numerically in `verify_P168.py`). $\square$

4 The theta series identity $\Theta_{E_8}(\tau) = E_4(\tau)$

4.1 Statement and classical proof sketch

Definition 4.1 (Weight-4 Eisenstein series). The *normalised Eisenstein series of weight 4* is

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad q = e^{2\pi i \tau}, \quad (6)$$

where $\sigma_3(n) = \sum_{d|n} d^3$ is the sum of cubes of divisors of n . It is a modular form of weight 4 for $\mathrm{SL}(2, \mathbb{Z})$.

Theorem 4.2 ($\Theta_{E_8} = E_4$). *The theta series of the E_8 lattice equals the Eisenstein series E_4 :*

$$\Theta_{E_8}(\tau) := \sum_{\lambda \in E_8} q^{|\lambda|^2/2} = E_4(\tau) = 1 + 240 q + 2160 q^2 + 6720 q^3 + 17520 q^4 + \dots \quad (7)$$

In particular, the coefficient of q^1 is $240 = |\Phi_{E_8}|$ (the root count), and E_4 is the unique weight-4 modular form for $\mathrm{SL}(2, \mathbb{Z})$ (up to scalar).

Proof. The proof proceeds in three steps.

Step 1: Modular weight. The theta series $\Theta_{E_8}(\tau) = \sum_{\lambda \in E_8} e^{\pi i |\lambda|^2 \tau}$ transforms as a modular form of weight $\text{rank}(E_8)/2 = 8/2 = 4$ under $\text{SL}(2, \mathbb{Z})$, by the Poisson summation formula applied to even self-dual lattices. Self-duality ($E_8^* = E_8$) ensures modular invariance under $\tau \mapsto -1/\tau$; evenness (all $|\lambda|^2 \in 2\mathbb{Z}$) ensures invariance under $\tau \mapsto \tau + 1$. Together these generate $\text{SL}(2, \mathbb{Z})$, so Θ_{E_8} is a modular form of weight 4 for the full modular group.

Step 2: Uniqueness. The space of modular forms of weight 4 for $\text{SL}(2, \mathbb{Z})$ is one-dimensional: $M_4(\text{SL}(2, \mathbb{Z})) = \mathbb{C} \cdot E_4$. Any two modular forms of weight 4 with the same leading Fourier coefficient (here, constant term 1) must be equal.

Step 3: Leading coefficient. The constant term of Θ_{E_8} is 1 (the single vector $\lambda = 0$), which matches the constant term of E_4 . Hence $\Theta_{E_8} = E_4$.

See [5, 4, 7] for detailed treatments. $\square$

Remark 4.3 (Why self-duality is essential). If E_8 were not self-dual, its theta series would transform as a modular form for a congruence subgroup of $\text{SL}(2, \mathbb{Z})$, not for the full group. Self-duality is exactly what lifts the theta series from a congruence-level form to a full-level form, and it is precisely at weight 4 that the full-level space is one-dimensional. The coincidence of $\text{rank}(E_8)/2 = 4$ with the unique non-trivial weight at which $M_\bullet(\text{SL}(2, \mathbb{Z}))$ is one-dimensional is what makes the identity $\Theta_{E_8} = E_4$ possible. No other self-dual lattice has this feature in the same way: the D_{16}^+ and $E_8 \oplus E_8$ lattices in $\mathbb{R}^{16}$ are both even self-dual but their weight-8 theta series are not equal (since $\dim M_8(\text{SL}(2, \mathbb{Z})) = 2$).

4.2 Fourier coefficients and the root count 240

Corollary 4.4 (Root count from theta series). *From Theorem 4.2:*

$$[q^1] \Theta_{E_8}(\tau) = 240 \cdot \sigma_3(1) = 240 \cdot 1 = 240 = |\Phi_{E_8}|. \quad (8)$$

More generally, $[q^n] \Theta_{E_8} = 240 \sigma_3(n)$, and the first few values are:

n	$\sigma_3(n)$	$[q^n] E_4$
1	1	240
2	$1 + 8 = 9$	2160
3	$1 + 27 = 28$	6720
4	$1 + 8 + 64 = 73$	17520

The coefficient 240 counts the roots of E_8 ; the coefficient 2160 counts the lattice vectors of squared norm 4 (the next shell after the roots).

Proof. $\sigma_3(1) = 1^3 = 1$, so $[q^1] E_4 = 240 \times 1 = 240$. The root count interpretation follows from the definition of the theta series: $[q^1] \Theta_{E_8}$ counts the lattice vectors $\lambda \in E_8$ with $|\lambda|^2/2 = 1$, i.e. $|\lambda|^2 = 2$. The shortest non-zero vectors of E_8 have squared norm 2, and there are exactly $|\Phi_{E_8}| = 240$ of them. $\square$

5 CM point evaluation at $\tau = i$

5.1 The identity $E_4(i) = J_{\text{short}} \cdot \eta(i)^8$

Theorem 5.1 (CM evaluation). *At the CM point $\tau = i$:*

$$\Theta_{E_8}(i) = E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8, \quad (9)$$

where $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$ (P159) and $J_{\text{short}} = 12$.

Proof. The identity $E_4(i) = J_{\text{short}} \cdot \eta(i)^8$ is established in P159 as a consequence of the classical relation between the j -function, the Eisenstein series E_4 , and the Dedekind eta function. We recall the derivation for completeness.

The j -invariant satisfies

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \quad \Delta(\tau) = \eta(\tau)^{24}, \quad (10)$$

so $E_4(\tau)^3 = j(\tau) \cdot \eta(\tau)^{24}$. At $\tau = i$, the CM value $j(i) = 1728 = J_{\text{short}}^3 = 12^3$ (established in P165 from P159) gives:

$$E_4(i)^3 = j(i) \cdot \eta(i)^{24} = 1728 \cdot \eta(i)^{24} = J_{\text{short}}^3 \cdot \eta(i)^{24}. \quad (11)$$

Taking real cube roots (both sides are positive real numbers since $\tau = i$ is a CM point with real values):

$$E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8. \quad (12)$$

Combined with $\Theta_{E_8}(i) = E_4(i)$ (Theorem 4.2), this yields the claimed identity. $\square$

Remark 5.2 (What the CM point measures). The TOE corpus uses $\tau = i$ as the reference CM point because it is the fixed point of $\tau \mapsto -1/\tau$ in the upper half-plane, i.e. the unique point of maximal symmetry under the S -transformation. At this point, the E_8 theta series becomes a pure number expressible in terms of the Gamma function:

$$\Theta_{E_8}(i) = 12 \cdot \left(\frac{\Gamma(1/4)}{2\pi^{3/4}} \right)^8.$$

Every factor is a TOE constant: the $12 = J_{\text{short}}$ is the central TOE integer, and $\Gamma(1/4)/\pi^{3/4}$ is the CM-period ratio. The self-dual structure of E_8 is what allows this clean collapse.

5.2 Numerical verification to 50 significant figures

The identity $\Theta_{E_8}(i) = E_4(i) = 12 \cdot \eta(i)^8$ is verified to 50 significant figures in `verify_P168.py` by two independent computations:

1. **q -expansion method:** evaluate $E_4(i)$ via $1 + 240 \sum_{n \geq 1} \sigma_3(n) \cdot e^{-2\pi n}$. With $q = e^{-2\pi} \approx 1.867 \times 10^{-3}$, the n -th term is $O(q^n)$; for 50 significant figures one requires $n > 50/\log_{10}(1/q) \approx 50/2.73 \approx 18.3$, so $n \leq 30$ suffices by a comfortable margin.
2. **Closed-form method:** compute $12 \cdot (\Gamma(1/4)/(2\pi^{3/4}))^8$ directly using `mpmath`'s gamma function at 55 digits of working precision.

The relative error between these two independent computations is bounded below 10^{-50} . The verification also checks (as assertion 9):

$$\frac{E_4(i)^3}{j(i)} = \eta(i)^{24} = \Delta(i), \quad (13)$$

confirming internal consistency with P164's $\Delta(i) = \eta(i)^{24}$.

6 E_8 in $j^{1/3}$: the first word

Theorem 6.1 ($\dim(E_8) = 248$ as the q^1 coefficient of $j^{1/3}$). *The McKay–Thompson function*

$$j(\tau)^{1/3} = q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \cdots), \quad q = e^{2\pi i\tau}, \quad (14)$$

has $[q^1]j^{1/3} = 248 = \dim(E_8)$ (established in P161).

Proof. The function $j^{1/3}(\tau) = q^{-1/3}(1 + 248q + \cdots)$ is a Hauptmodul for a certain genus-0 modular curve, and its coefficients are related to the McKay–Thompson series for the Monster group. That the leading non-trivial coefficient is $248 = \dim(E_8)$ was first observed by McKay and is part of the Moonshine programme; the derivation is given in P161. Here we note the structural implication: $j^{1/3}$ “knows” the dimension of E_8 before it knows anything else. The next coefficient $4124 = 4096 + 27 + 1 = 2^{12} + 27 + 1$ and higher coefficients encode dimensions of E_8 representations. $\square$

Remark 6.2 (The first word of the j -function). In the expansion $j^{1/3}(\tau) = q^{-1/3}(1 + 248q + \cdots)$, the leading term $q^{-1/3}$ is the singularity at the cusp; it carries no algebraic content. The number 1 (constant term) is the empty representation. The number 248 is the adjoint representation of E_8 . In this sense, E_8 is literally the first algebraic content of the j -function: the first word after the empty word. This is part of what justifies the OM designation in the TOE programme.

Proposition 6.3 (Connection to the E_8 root system). *The q^1 coefficient $248 = \dim(E_8)$ decomposes as $8 + 240$:*

$$248 = \underbrace{8}_{\text{Cartan}} + \underbrace{240}_{\text{roots}}, \quad (15)$$

reflecting the decomposition of $\mathfrak{e}_8$ as a vector space into the Cartan subalgebra (8-dimensional) and the root spaces (one for each of the 240 roots). The $j^{1/3}$ coefficient counts the adjoint representation, which is the Lie algebra itself.

Proof. The adjoint representation of E_8 is $\mathfrak{e}_8$ itself, of dimension $248 = \text{rank}(E_8) + |\Phi_{E_8}| = 8 + 240$ (Equation (4)). The identification of $[q^1]j^{1/3}$ with $\dim(\text{adj})$ is a consequence of the McKay–Thompson theory (P161). $\square$

7 The gauge cascade: $E_8 \rightarrow G_2$

7.1 Root sub-systems and the breaking chain

Theorem 7.1 (The symmetry-breaking cascade). *The exceptional chain $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ (strict containments as root sub-systems) has root counts:*

$$12 \subset 48 \subset 72 \subset 126 \subset 240. \quad (16)$$

At each step, the new roots are:

$$|\Phi_{F_4}| - |\Phi_{G_2}| = 48 - 12 = 36 = B_{F_4}, \quad (17)$$

$$|\Phi_{E_6}| - |\Phi_{F_4}| = 72 - 48 = 24, \quad (18)$$

$$|\Phi_{E_7}| - |\Phi_{E_6}| = 126 - 72 = 54 = 27 + 27, \quad (19)$$

$$|\Phi_{E_8}| - |\Phi_{E_7}| = 240 - 126 = 114. \quad (20)$$

Proof. The inclusions of root systems $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ are classical (see [3, 8]). The root counts are established in P151–P167 for G_2, F_4, E_6, E_7 and in Theorem 2.1 for E_8 .

The increment $|\Phi_{F_4}| - |\Phi_{G_2}| = 36 = B_{F_4}$ was the central result of P165. The increment $54 = 27 + 27$ matches the complement splitting proved in P167, Proposition 5.2: the 54 roots in

$\Phi_{E_7} \setminus \Phi_{E_6}$ decompose as $\mathbf{27} + \overline{\mathbf{27}}$ under E_6 . The final increment $114 = 240 - 126$ is the number of roots gained in the step $E_7 \rightarrow E_8$; these encode the E_7 -representation content of $\mathfrak{e}_8/\mathfrak{e}_7$ and are related to the 56-dimensional Freudenthal system: $114 = 2 \times 57 = 2 \times (56 + 1)$, where the factor of 2 is the $\pm$ root pairing. $\square$

Remark 7.2 (The SM gauge group in the cascade). The Standard Model gauge group $G_{\text{SM}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ embeds in the cascade: $G_{\text{SM}} \subset G_2 \subset E_8$ (through the G_2 sector, which contains $\text{SU}(3)$ as a maximal sub-algebra). A more direct embedding route runs $G_{\text{SM}} \subset E_6$ (via the $SO(10)$ route of P166) $\subset E_8$. The cascade $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow \dots$ is thus a symmetry-breaking cascade, and the 240 roots of E_8 are the “mother root system” containing all the root structures seen in the Standard Model. The relation between the cascade and actual symmetry breaking (i.e. which sub-root-systems survive after spontaneous symmetry breaking) depends on the vacuum structure and is not determined by the Lie algebra embedding alone.

7.2 Coxeter numbers across the cascade

Proposition 7.3 (Coxeter chain arithmetic). *The Coxeter numbers satisfy:*

$$h(E_8) = h(E_6) + h(E_7) = 12 + 18 = 30, \quad (21)$$

$$h(E_8) = 5 \times h(G_2) = 5 \times 6, \quad (22)$$

$$h(E_8)/h(E_7) = 30/18 = 5/3, \quad (23)$$

$$h(E_8)/h(E_6) = 30/12 = 5/2. \quad (24)$$

The complete chain of Coxeter numbers, Coxeter planes, and orbit counts is:

Algebra	h	Coxeter plane	Orbits	$ \Phi /h$
G_2	6	6-fold	2	2
F_4	12	12-fold	4	4 (two radii)
E_6	12	12-fold	6	6
E_7	18	18-fold	7	7
E_8	30	30-fold	8	8

For simply-laced algebras the orbit count equals the rank; F_4 is the exception (two root lengths, hence four orbits at two radii).

Proof. Equation (21): proved as part of Theorem 2.3. Equation (22): $5 \times 6 = 30$ (elementary). The table entries follow from the respective theorems in P164–P167 and Theorem 2.5 of this paper. $\square$

8 Conclusion: what E_8 contributes before Leech and Monster

The results of this addendum are collected in the following summary:

$$\begin{aligned}
h(E_8) &= h^\vee(E_8) = 30 = h(E_6) + h(E_7) = 5 \times 6 \\
|\Phi_{E_8}| &= 240 = 8 \times 30 = \text{rank}(E_8) \times h(E_8) \\
\dim(E_8) &= 248 = 8 + 240 = [q^1] j^{1/3}(\tau) \\
E_8^* &= E_8 \quad (\text{unique even self-dual lattice in } \mathbb{R}^8, \text{ OM}) \\
\Theta_{E_8}(\tau) &= E_4(\tau) \quad (\text{classical identity}) \\
\Theta_{E_8}(i) &= E_4(i) = J_{\text{short}} \cdot \eta(i)^8 = 12 \cdot \eta(i)^8 \\
E_4(i)^3/j(i) &= \eta(i)^{24} = \Delta(i) \\
12 &\subset 48 \subset 72 \subset 126 \subset 240 \quad (\text{root chain } G_2 \subset \dots \subset E_8)
\end{aligned} \quad (25)$$

The central thread of this addendum: E_8 is the unique algebra in the exceptional chain for which the theta series of its root lattice equals a full-level Eisenstein series. This is made possible by three coincidences that all occur simultaneously at E_8 : (1) $\text{rank}/2 = 4$ is the unique weight at which $M_\bullet(\text{SL}(2, \mathbb{Z}))$ is one-dimensional; (2) the lattice is self-dual, which lifts the theta series to the full modular group; (3) the lattice is even, ensuring the theta series lies in the correct subspace. None of G_2, F_4, E_6, E_7 satisfy all three simultaneously; only E_8 does.

What E_8 contributes before Leech and Monster.

1. *The self-dual anchor.* E_8 establishes the first self-dual lattice in the chain. The Leech lattice Λ_{24} is also even self-dual, but in $\mathbb{R}^{24}$; it is constructed from E_8 (via the Niemeier lattice E_8^3 and the “holy construction”, see [4]). Without E_8 , the Leech lattice has no algebraic root.
2. *The Eisenstein bridge.* The identity $\Theta_{E_8} = E_4$ connects the root system geometry to the ring of modular forms. It is this bridge that eventually becomes the Monstrous Moonshine connection (Monster characters $\leftrightarrow$ modular functions), and E_8 is the first algebra in the chain to cross it.
3. *The j -function first word.* The coefficient $248 = [q^1]j^{1/3}$ makes E_8 the first algebraic data encoded in the j -function. The Monster is the “ j -function’s symmetry group”; E_8 is what the j -function says first.
4. *The complete gauge cascade.* The symmetry-breaking chain $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow F_4 \rightarrow G_2$ is now fully documented in this corpus (P151–P168), with root counts, Coxeter numbers, and representation content at each step.

Open directions for P169 and beyond.

1. *The Leech lattice.* Addendum 169 should construct the Leech lattice Λ_{24} from E_8 (via the Niemeier lattice $E_8 \oplus E_8 \oplus E_8$ construction or the “Golay code” route) and establish $|\Lambda_{24}^{\min}| = 196560$ (the shortest-vector count), connecting to the Monster.
2. *$E_8 \subset \text{Monster}$.* The Monster sporadic group $\mathbb{M}$ contains E_8 as part of its structure (through the Griess algebra and Vertex Operator Algebra). The precise embedding $E_8 \hookrightarrow \mathbb{M}$ and the dimension count $196884 = 248 + 196636$ (McKay’s observation) is the next step.
3. *TOE operator $\hat{O}$ and E_8 .* The Wheel operators Fork/Weld/Plateau/Oscillate/Perturb of Paper 18 act on S^3 -valued quaternion states. Whether the 240-root orbit of E_8 can be realised as a trajectory in the Wheel’s S^3 geometry — i.e. whether E_8 is a fixed-point attractor of the Wheel dynamics — is an open computation connecting the algebraic chain to the dynamical kernel.
4. *Self-duality and the Butlerian line.* The property $E_8^* = E_8$ (self-dual, primordial) is the algebraic analogue of the TOE’s no-substitution principle: the structure accepts no external completion. Formalising this analogy — making precise in what sense the Wheel kernel is the dynamical realisation of the E_8 self-dual structure — is a long-horizon goal of the programme.

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