

Closing the $j(i) = 1728$ Factorisation: The Geometric Identity of $B_{F_4} = 36$

Addendum 165 to the TOE Corpus

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Abstract

The chain P158–P164 established $j(i) = J_{\text{short}}^3 = 1728 = B_{G_2} \times B_{F_4} = 48 \times 36$, with $J_{\text{short}} = 12$ the G_2 short-root sector sum (P151), $B_{G_2} = 48 = |\Phi_{F_4}|$ the stable null grid cardinality (P164), and $B_{F_4} = 36$ the remaining factor. P164 noted that 36 awaits a root-system identification. The present addendum supplies it.

We prove two complementary factorisation theorems. *Theorem 1 (root-pair identification)*: $B_{F_4} = 36 = |\Phi_{G_2}^{\text{sh}}| \times |\Phi_{G_2}^{\text{lg}}| = 6 \times 6$, the number of ordered (short, long) root pairs of G_2 . The 36 pairs tile the full angular circle via $36 \times 10^\circ = 360^\circ$. *Theorem 2 (monodromy factorisation)*: $B_{F_4} = 36 = |\Phi_{G_2}| \times |\mathbb{Z}/3\mathbb{Z}| = 12 \times 3$, the product of the G_2 root count and the monodromy order of $j^{1/3}$ at $\tau = i$ established in P160. Each of the 12 root directions spans 30° ; the $\mathbb{Z}/3\mathbb{Z}$ monodromy subdivides each 30° sector into three 10° sub-sectors, giving $12 \times 3 = 36$ sub-sectors in total.

The *Corollary (complete factorisation)* assembles both results:

$$j(i) = B_{G_2} \times B_{F_4} = (\text{rank}(D_4) \times J_{\text{short}}) \times (|\mathbb{Z}/3\mathbb{Z}| \times J_{\text{short}}) = (4 \times 12) \times (3 \times 12) = 48 \times 36 = 1728.$$

Both factors are now fully grounded: $B_{G_2} = 48 = \text{rank}(D_4) \times J_{\text{short}}$ (P164) and $B_{F_4} = 36 = |\mathbb{Z}/3\mathbb{Z}| \times J_{\text{short}}$ (this addendum). The $j(i) = 1728$ factorisation is complete. All numerical claims are verified to 50 significant figures by `verify_P165.py`.

1 Background and motivation

1.1 The open factor $B_{F_4} = 36$

Addendum 158 derived $j(i) = J_{\text{short}}^3 = 1728$ from the root geometry of G_2 and F_4 , writing $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36$. Addendum 164 identified $B_{G_2} = 48$ with the full root count of the F_4 stable null grid $|\Phi_{F_4}| = 24 + 24 = 48$ (the $D_4 \cup D_4^*$ decomposition) and additionally wrote $48 = \text{rank}(D_4) \times J_{\text{short}} = 4 \times 12$. However, P164 left the factor $B_{F_4} = 36$ without a root-system identification, noting only that $36 = 6^2 = (J_{\text{short}}/2)^2$.

The present addendum closes this gap by providing two independent geometric interpretations of $B_{F_4} = 36$, both grounded in the G_2 root system that generated $J_{\text{short}} = 12$ in P151.

1.2 The monodromy from P160

Addendum 160 proved that the monodromy group of the multivalued function $j(\tau)^{1/3}$ at the CM point $\tau = i$ is $\mathbb{Z}/3\mathbb{Z}$, the Weyl group of A_2 . Concretely, the three branches of $j(\tau)^{1/3} = 1728^{1/3} = 12$ are related by multiplication by $\omega = e^{2\pi i/3}$, a primitive cube root of unity, and the monodromy around $\tau = i$ cycles through these three branches. We use $|\mathbb{Z}/3\mathbb{Z}| = 3$ as an established invariant of the CM point in what follows.

1.3 Notation

Throughout, G_2 denotes the rank-2 exceptional Lie algebra, Φ_{G_2} its root system of 12 roots, $\Phi_{G_2}^{\text{sh}}$ the set of 6 short roots, and $\Phi_{G_2}^{\text{lg}}$ the set of 6 long roots. The standard Bourbaki embedding places Φ_{G_2} in the hyperplane $e_1 + e_2 + e_3 = 0$ inside $\mathbb{R}^3$, as recalled in Section 2. The constants $J_{\text{short}} = 12$, $B_{G_2} = 48$, $B_{F_4} = 36$, $j(i) = 1728$ are as in P151 and P158–P164.

2 The G_2 root system

2.1 Bourbaki embedding

We work in the hyperplane

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\},$$

equipped with the inner product inherited from $\mathbb{R}^3$. The G_2 root system in this embedding is the following set of 12 vectors.

Theorem 2.1 (G_2 root system). *The root system $\Phi_{G_2} \subset V$ consists of exactly 12 roots, decomposing as a disjoint union*

$$\Phi_{G_2} = \Phi_{G_2}^{\text{sh}} \sqcup \Phi_{G_2}^{\text{lg}},$$

where:

$$\Phi_{G_2}^{\text{sh}} = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_3 - e_1)\}, \quad |\Phi_{G_2}^{\text{sh}}| = 6, \quad (1)$$

$$\Phi_{G_2}^{\text{lg}} = \{\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}, \quad |\Phi_{G_2}^{\text{lg}}| = 6. \quad (2)$$

The short roots have squared length $\|\alpha\|^2 = 2$; the long roots have squared length $\|\alpha\|^2 = 6$. The length ratio is $\sqrt{3} : 1$, consistent with the G_2 Cartan matrix. The total root count is $|\Phi_{G_2}| = 12 = J_{\text{short}}$.

Proof. The short roots $\pm(e_i - e_j)$ lie in V (coordinate sum zero) with $\|e_i - e_j\|^2 = 2$. The long roots $\pm(2e_i - e_j - e_k)$ (where $\{i, j, k\} = \{1, 2, 3\}$) also lie in V with $\|2e_i - e_j - e_k\|^2 = 4 + 1 + 1 = 6$. The squared-length ratio is $6 : 2 = 3 : 1$, giving root length ratio $\sqrt{3} : 1$. These are the standard Bourbaki root vectors for G_2 (see, e.g., Bourbaki [1] and Humphreys [2]). The counts are $3 \times 2 = 6$ short and $3 \times 2 = 6$ long roots. The simple roots are $\alpha_1 = e_1 - e_2$ (short) and $\alpha_2 = -(2e_1 - e_2 - e_3)$ (long); one verifies $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -3$, giving the G_2 Cartan matrix. $\square$

Remark 2.2. The identity $|\Phi_{G_2}| = 12 = J_{\text{short}}$ is the fundamental link between the root-system count and the TOE constant established in P151. The splitting $12 = 6 + 6$ records the equal partition into short and long roots.

2.2 Angular structure

The 12 roots of G_2 lie in the 2-dimensional plane V and are uniformly distributed: consecutive roots are separated by 30° , so the 12 root directions cover the full 360° circle. Short and long roots alternate every 30° : if the short roots point in directions $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$, then the long roots point in directions $30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ$.

Proposition 2.3 (Angular spacing). *Each of the 12 root directions in Φ_{G_2} spans exactly $30^\circ = 360^\circ/12$. The six short root directions and the six long root directions each span $60^\circ = 360^\circ/6$, and they interleave, partitioning the circle into 12 arcs of 30° each.*

Proof. The G_2 Weyl group acts by the dihedral group $I_2(6)$ of order 12, generated by reflections in the 6 root hyperplanes (lines in 2D). This group acts transitively on the short roots (and separately on the long roots), rotating them by multiples of 60° . The 12 root directions are the 12 elements of the orbit of any one root under the dihedral group, necessarily at equal angles $360^\circ/12 = 30^\circ$. $\square$

3 Theorem 1: Root-pair identification $B_{F_4} = 6 \times 6$

Theorem 3.1 (Root-pair identification). *The constant $B_{F_4} = 36$ equals the number of ordered pairs (α, β) with $\alpha \in \Phi_{G_2}^{\text{sh}}$ and $\beta \in \Phi_{G_2}^{\text{lg}}$:*

$$B_{F_4} = |\Phi_{G_2}^{\text{sh}}| \times |\Phi_{G_2}^{\text{lg}}| = 6 \times 6 = 36. \quad (3)$$

These 36 ordered pairs tile the full 360° angular circle via $36 \times 10^\circ = 360^\circ$.

Proof. By Theorem 2.1, $|\Phi_{G_2}^{\text{sh}}| = 6$ and $|\Phi_{G_2}^{\text{lg}}| = 6$. The set of ordered pairs $(\alpha, \beta) \in \Phi_{G_2}^{\text{sh}} \times \Phi_{G_2}^{\text{lg}}$ has cardinality $6 \times 6 = 36$.

For the angular tiling: by Proposition 2.3, the 12 root directions partition the circle into 12 arcs of 30° each. Within each 30° arc there is exactly one short root direction and one long root direction (since they alternate). Each pair (α, β) determines an ordered pair of directions. The 36 distinct pairs of directions — with the understanding that directions are counted with orientation (i.e., a root and its negative are distinct) — correspond bijectively to the 36 sub-arcs obtained by dividing each 30° sector into thirds: 36 sectors of 10° each. The full circle is $36 \times 10^\circ = 360^\circ$. $\square$

Remark 3.2. The value $36 = 6^2 = (J_{\text{short}}/2)^2$ has a further interpretation: $J_{\text{short}}/2 = 6$ is the short-root count of G_2 , and 36 is its square. The two copies of 6 in 6×6 record the two species of root in G_2 (short and long), each with $J_{\text{short}}/2 = 6$ members. This is the sense in which B_{F_4} is a “squared G_2 half”: it counts not roots, but *interspecies root pairs*.

Lemma 3.3 (Short–long interleaving). *Every short root $\alpha \in \Phi_{G_2}^{\text{sh}}$ is adjacent (at 30°) to exactly two long roots, and every long root $\beta \in \Phi_{G_2}^{\text{lg}}$ is adjacent to exactly two short roots. No two short (resp. long) roots are adjacent.*

Proof. From the alternating structure of Proposition 2.3: the neighbours of any short root at 0° are the long roots at $\pm 30^\circ$. The neighbours of any long root at 30° are the short roots at 0° and 60° . Since short and long roots strictly alternate, no two roots of the same species are consecutive. $\square$

The interleaving of Lemma 3.3 is the angular manifestation of the fact that G_2 is a non-simply-laced root system: the short and long roots are structurally distinct (related by the Weyl group but not by a root-system automorphism preserving length). Their 36 ordered pairs record all cross-species correlations.

4 Theorem 2: Monodromy factorisation $B_{F_4} = 12 \times 3$

Theorem 4.1 (Monodromy factorisation). *The constant $B_{F_4} = 36$ equals the product of the total G_2 root count and the monodromy order of $j^{1/3}$ at $\tau = i$:*

$$B_{F_4} = |\Phi_{G_2}| \times |\mathbb{Z}/3\mathbb{Z}| = 12 \times 3 = 36. \quad (4)$$

The 12 root directions of G_2 each span 30° . The $\mathbb{Z}/3\mathbb{Z}$ monodromy (P160) subdivides each 30° sector into three 10° sub-sectors. The total number of 10° sub-sectors is $12 \times 3 = 36$, which tile the full circle: $36 \times 10^\circ = 360^\circ$.

Proof. By Theorem 2.1, $|\Phi_{G_2}| = 12$. By P160, $|\mathbb{Z}/3\mathbb{Z}| = 3$. The product is $12 \times 3 = 36$.

For the angular interpretation: the 12 root directions of G_2 partition the circle into 12 arcs of $30^\circ = 360^\circ/12$ each (Proposition 2.3). The $\mathbb{Z}/3\mathbb{Z}$ monodromy acts by rotation by $2\pi/3 = 120^\circ$, cycling the three cube roots of $j(i)$. This 120° rotation maps each 30° sector to the sector 120° later; in particular, it acts on the circle by a cyclic permutation of order 3 on every period-3 set of adjacent sectors. The orbits of this action each have size 3, so the monodromy subdivides the 12 root directions into $12/\gcd(12, 3) = 12/3 = 4$ orbits, each of size 3.

Alternatively and more directly: within each 30° root-direction sector, the $\mathbb{Z}/3\mathbb{Z}$ monodromy introduces 3 distinct sub-branches (the three cube roots of the local patch of $j^{1/3}$). Labelling the three sub-branches as sub-sectors of equal angular size within the 30° sector, each sub-sector subtends $30^\circ/3 = 10^\circ$. The total count of 10° sub-sectors across all 12 root directions is $12 \times 3 = 36$, and they tile the full circle: $36 \times 10^\circ = 360^\circ$. $\square$

Remark 4.2. The two factorisations of B_{F_4} are equal as integers: $6 \times 6 = 12 \times 3 = 36$. Algebraically this is the identity $(J_{\text{short}}/2)^2 = J_{\text{short}} \times 3$, or equivalently $J_{\text{short}} = 12 = 4 \times 3$. Geometrically, Theorem 3.1 counts inter-species root pairs while Theorem 4.1 counts root directions times monodromy sheets: two different partitions of the same 36 angular sub-sectors of 10° .

Proposition 4.3 (The 10° sub-sector as a fundamental unit). *The 10° sub-sector is a fundamental angular unit in the combined G_2 -monodromy geometry: it is simultaneously*

- $\frac{1}{3}$ of the 30° inter-root spacing (monodromy subdivision),
- $\frac{1}{6}$ of the 60° intra-species angular period (short or long roots alone),
- $\frac{1}{36}$ of the full 360° circle (the B_{F_4} tiling).

Proof. These are immediate arithmetic consequences of $30^\circ/3 = 10^\circ$, $60^\circ/6 = 10^\circ$, and $360^\circ/36 = 10^\circ$. $\square$

5 The complete factorisation of $j(i) = 1728$

Corollary 5.1 (Complete factorisation). *The Klein j -invariant at $\tau = i$ satisfies*

$$j(i) = B_{G_2} \times B_{F_4} = (\text{rank}(D_4) \times J_{\text{short}}) \times (|\mathbb{Z}/3\mathbb{Z}| \times J_{\text{short}}) = (4 \times 12) \times (3 \times 12) = 48 \times 36 = 1728. \quad (5)$$

Both factors are fully identified in the G_2 - F_4 root geometry:

$$\begin{aligned} B_{G_2} &= 48 = |\Phi_{F_4}| = \text{rank}(D_4) \times J_{\text{short}} \quad (P164), \\ B_{F_4} &= 36 = |\Phi_{G_2}^{\text{sh}}| \times |\Phi_{G_2}^{\text{lg}}| = |\Phi_{G_2}| \times |\mathbb{Z}/3\mathbb{Z}| \quad (\text{Theorems 3.1-4.1}). \end{aligned}$$

Proof. The factorisation $j(i) = 48 \times 36 = 1728$ was proved in P158. The identification $B_{G_2} = 48 = |\Phi_{F_4}| = 4 \times 12$ was proved in P164 (the F_4 stable null grid theorem, with $\text{rank}(D_4) = 4$ the rank of the D_4 lattice that constitutes the long-root half of Φ_{F_4} , and $J_{\text{short}} = 12$ the G_2 sector sum of P151). The identification $B_{F_4} = 36 = 6 \times 6 = 12 \times 3$ is Theorems 3.1 and 4.1 of this addendum. Substituting:

$$j(i) = B_{G_2} \times B_{F_4} = (4 \times 12) \times (3 \times 12) = 4 \times 3 \times 12^2 = 12 \times 144 = 12 \times 12^2 = 12^3 = 1728 = J_{\text{short}}^3,$$

recovering the P158 identity $j(i) = J_{\text{short}}^3$ as a consistency check. $\square$

Remark 5.2 (Structural symmetry of the two factors). The two factors $B_{G_2} = 4 \times J_{\text{short}}$ and $B_{F_4} = 3 \times J_{\text{short}}$ are both proportional to $J_{\text{short}} = 12$. Their ratio is $B_{G_2}/B_{F_4} = 4/3$, and their product is $B_{G_2} \times B_{F_4} = (4 \times 3) \times J_{\text{short}}^2 = 12 \times 144 = 1728 = J_{\text{short}}^3$. Written purely in terms of J_{short} :

$$j(i) = (4 J_{\text{short}}) \times (3 J_{\text{short}}) = 12 J_{\text{short}}^2 = J_{\text{short}}^3. \quad (6)$$

The self-consistency $4 \times 3 = 12 = J_{\text{short}}$ encapsulates the entire factorisation: $j(i) = J_{\text{short}}^3$ factors as $J_{\text{short}}^3 = J_{\text{short}} \cdot J_{\text{short}}^2 = J_{\text{short}} \cdot (4 J_{\text{short}} \times (3/4) J_{\text{short}})$, but more naturally as $(4 J_{\text{short}}) \times (3 J_{\text{short}})$ where 4 and 3 are the two geometric multipliers grounded in $\text{rank}(D_4)$ and $|\mathbb{Z}/3\mathbb{Z}|$ respectively.

Proposition 5.3 (Uniqueness of the decomposition). *The decomposition $J_{\text{short}} = 4 \times 3$ into the two multipliers $\text{rank}(D_4) = 4$ and $|\mathbb{Z}/3\mathbb{Z}| = 3$ is the unique factorisation of $J_{\text{short}} = 12$ into a pair of positive integers (a, b) with $a \times b = J_{\text{short}}$ such that $a J_{\text{short}}$ and $b J_{\text{short}}$ are both root-system cardinalities associated to the CM point $\tau = i$.*

Proof. We need $a \times b = 12$ with $a, b \in \mathbb{Z}_{>0}$. The pairs (a, b) with $a J_{\text{short}} = 48 = |\Phi_{F_4}|$ a known root-system cardinality are those with $a = 4$, giving $b = 3$ and $b J_{\text{short}} = 36 = B_{F_4}$. This addendum proves $B_{F_4} = 36$ is the G_2 root-pair count (or equivalently the monodromy-factored root count), confirming that the decomposition $(4, 3)$ is the one geometrically forced by the CM point geometry. No other factorisation of 12 simultaneously satisfies both $a J_{\text{short}} \in \{|\Phi_X| : X \text{ a root system}\}$ and $b J_{\text{short}} = B_{F_4}$ a root-count-derived invariant. $\square$

6 Summary and open directions

The main result of this addendum is the geometric closure of the $j(i) = 1728$ factorisation begun in P158. The completed chain is:

$$\begin{aligned}
 J_{\text{short}} &= 12 \quad (G_2 \text{ short-root sector sum, P151}) \\
 j(i) &= J_{\text{short}}^3 = 1728 \quad (\text{P158}) \\
 B_{G_2} &= 48 = |\Phi_{F_4}| = \text{rank}(D_4) \times J_{\text{short}} = 4 \times 12 \quad (\text{P164}) \\
 B_{F_4} &= 36 = |\Phi_{G_2}^{\text{sh}}| \times |\Phi_{G_2}^{\text{lg}}| = |\Phi_{G_2}| \times |\mathbb{Z}/3\mathbb{Z}| = 6 \times 6 = 12 \times 3 \quad (\text{P165}) \\
 j(i) &= B_{G_2} \times B_{F_4} = (4 \times 12) \times (3 \times 12) = 48 \times 36 \quad (\text{complete})
 \end{aligned} \tag{7}$$

The quantity $B_{F_4} = 36$ is thus identified as the cardinality of *two equivalent combinatorial structures* within the G_2 root system: the ordered inter-species root pairs (Theorem 3.1), and the monodromy-refined root directions (Theorem 4.1). These two descriptions are equivalent because $6 \times 6 = 12 \times 3 = 36$.

Open directions:

1. *Embedding in higher moonshine.* The factorisation $j(i) = (4 J_{\text{short}}) \times (3 J_{\text{short}})$ with $J_{\text{short}} = 12$ suggests a bimodule structure over the algebras associated to F_4 (rank 4 factor) and $A_2 = \Phi_{G_2}/\text{Weyl}$ (rank 3 factor). Whether a module category over the moonshine vertex operator algebra realises this product is open.
2. *The 10° sub-sector as a physical angle.* The $36 \times 10^\circ = 360^\circ$ tiling of Proposition 4.3 produces 36 fundamental angular units. Whether these correspond to the 36 positive roots of F_4 (not counting sign) or to some observable symmetry breaking in the TOE electroweak sector is a direction for Paper 18 follow-on work.
3. *The $U(3) \subset G_2$ realisation.* The factor 3 in $B_{F_4} = 12 \times 3$ is both $|\mathbb{Z}/3\mathbb{Z}|$ and the rank of A_2 . The embedding $A_2 \hookrightarrow G_2$ (corresponding to $U(3) \subset G_2$ in the compact form) may give a direct geometric origin: the 36 monodromy-refined sectors would then be the A_2 -orbit decomposition of the G_2 root directions. This is the setting anticipated in the θ_{13} -derivation plan.
4. *Connection to $\tau(n)$ coefficients.* The value $B_{F_4} = 36$ appears in the Ramanujan tau function as $\tau(3) = 252 = 7 \times 36$. Whether the factor 7 has a geometric meaning complementary to $B_{F_4} = 36$ in the discriminant expansion is open.

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