

The Stable Null Grid: F_4 Root Decomposition, the Ramanujan Discriminant, and Chiral Cancellation at the CM Point $\tau = i$

Addendum 164 to the TOE Corpus

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Erratum (A343, 2026-06-15): The provenance line “ $j(i) = J_{\text{short}}^3$ was *proved using* the McKay-Thompson identification $T_{3B}(i) = J_{\text{short}}$ (A163)” rests on the conflation retracted by A337: $j^{1/3}(i) = 12 = J_{\text{short}}$ is the E8/G2 cube-root contact, *not* a Monster 3B McKay-Thompson value. The fact $j(i) = J_{\text{short}}^3 = 1728$ stands — it was independently derived in A158 from G_2/F_4 root geometry — as do $\Delta(i) = \eta(i)^{24}$ and the $F_4 = D_4 \cup D_4^*$ decomposition. Only the “proved using T_{3B} ” attribution is withdrawn. See A337/A343. Body preserved as audit trail.

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Abstract

The chain P158–P163 established $J_{\text{short}} = 12$, $j(i) = 1728 = B_{G_2} \times B_{F_4}$ with $B_{G_2} = 48$, and identified $\tau = i$ as the unique CM point where Monster moonshine contacts TOE geometry. The present addendum asks what algebraic object $B_{G_2} = 48$ is, and why the Ramanujan discriminant $\Delta(\tau) = \eta(\tau)^{24}$ carries exponent $24 = B_{G_2}/2$ rather than B_{G_2} itself.

We prove: (i) the F_4 root system decomposes as $\Phi_{F_4} = \Phi_{\text{long}} \cup \Phi_{\text{short}}$ where Φ_{long} (24 roots) is the D_4 root lattice and Φ_{short} (24 roots) is the minimal vectors of D_4^* , the lattice dual to D_4 ; (ii) each set of 24 roots realises the vertices of the 24-cell, the unique self-dual regular polytope in four dimensions; (iii) the two chiral halves are related by lattice duality — neither is heavier than the other, making Φ_{F_4} the *stable null grid*; (iv) the exponent 24 in $\Delta(\tau) = \eta(\tau)^{24}$ counts the roots of one chiral half (D_4 or D_4^*), and the Ramanujan tau coefficient $\tau(2) = -24$ is the chiral-cancellation signature at first order; (v) the identity $\Delta(i) = E_4(i)^3/j(i) = \eta(i)^{24}$ expresses the discriminant at the CM point entirely in the TOE quantities $J_{\text{short}} = 12$ and $j(i) = J_{\text{short}}^3 = 1728$ established in P158–P159; and (vi) $B_{G_2} = 48 = |\Phi_{F_4}|$ is the full stable null grid cardinality, so $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36$ is the product of the grid size and a complementary branching factor. All numerical claims are verified to 50 significant figures by `verify_P164.py`.

1 Background and motivation

1.1 The constant $B_{G_2} = 48$ and the question it raises

Addendum 158 derived $j(i) = J_{\text{short}}^3 = 1728$ from the root geometry of G_2 and F_4 , writing $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36$. The quantity $J_{\text{short}} = 12$ was identified as the G_2 short-root sector sum (Addendum 151), and $j(i) = J_{\text{short}}^3$ was proved using the McKay–Thompson identification $T_{3B}(i) = J_{\text{short}}$ (Addendum 163). What remained unidentified was $B_{G_2} = 48$ itself: it is not the root count of G_2 (which is 12), nor any obviously canonical invariant of G_2 alone.

The identification becomes clear one step up: F_4 has exactly 48 roots. The subscript G_2 in B_{G_2} refers to the G_2 subalgebra of F_4 , and the value 48 is the full F_4 root count. The present addendum unpacks the internal structure of those 48 roots and shows that it carries a precise chiral-duality interpretation connecting $\Delta(\tau)$ to the CM point.

1.2 The Ramanujan discriminant

The Ramanujan discriminant is the unique normalised weight-12 cusp form for $\text{SL}(2, \mathbb{Z})$:

$$\Delta(\tau) = \eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i \tau}, \quad (1)$$

where $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function. The Fourier expansion is

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n, \quad (2)$$

where $\tau(n)$ denotes the Ramanujan tau function. The first several values are

$$\tau(1) = 1, \quad \tau(2) = -24, \quad \tau(3) = 252, \quad \tau(4) = -1472, \quad \tau(5) = 4830, \quad \tau(6) = -6048. \quad (3)$$

The exponent 24 in (1) and the coefficient $\tau(2) = -24$ are both determined by the single integer 24. The main question of this addendum is: *what does 24 count in the TOE geometry?*

2 The F_4 root system as a stable null grid

2.1 Root system of F_4

We work in $\mathbb{R}^4$ with the standard inner product $\langle e_i, e_j \rangle = \delta_{ij}$.

Theorem 2.1 (F_4 root decomposition). *The root system Φ_{F_4} consists of exactly 48 vectors, decomposing as a disjoint union*

$$\Phi_{F_4} = \Phi_{\text{long}} \sqcup \Phi_{\text{short}},$$

where:

$$\Phi_{\text{long}} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}, \quad |\Phi_{\text{long}}| = 24, \quad (4)$$

$$\Phi_{\text{short}} = \{\pm e_i \mid i = 1, 2, 3, 4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}, \quad |\Phi_{\text{short}}| = 24. \quad (5)$$

Each set has cardinality 24; the total is $|\Phi_{F_4}| = 48$.

Proof. These are the standard root vectors of F_4 in the Bourbaki normalisation (see, e.g., Humphreys [1]). The long roots $\pm e_i \pm e_j$ have squared length $\|\pm e_i \pm e_j\|^2 = 2$; the short roots $\pm e_i$ have squared length 1 and the half-diagonal roots $\frac{1}{2}(\pm 1)^4$ have squared length 1. The ratio of squared lengths is $2 : 1 = (\sqrt{2} : 1)^2$, consistent with the Cartan matrix of F_4 . The counts: $\binom{4}{2} \times 4 = 24$ long roots; $2 \times 4 + 2^4 = 8 + 16 = 24$ short roots. $\square$

2.2 The D_4 lattice and its dual

Definition 2.2 (D_4 and D_4^* lattices). The D_4 root lattice is

$$D_4 = \{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \mid n_1 + n_2 + n_3 + n_4 \equiv 0 \pmod{2}\}.$$

Its minimal vectors (squared length 2) are exactly Φ_{long} . The dual lattice D_4^* is

$$D_4^* = \{x \in \mathbb{R}^4 \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in D_4\} = \mathbb{Z}^4 \cup \left(\mathbb{Z}^4 + \frac{1}{2}(1, 1, 1, 1)\right).$$

The minimal vectors of D_4^* (squared length 1) are exactly Φ_{short} .

Theorem 2.3 (Lattice duality of chiral halves).

$$\Phi_{\text{short}} = \{\alpha \in D_4^* \mid \|\alpha\|^2 = 1\} \quad \text{and} \quad \Phi_{\text{long}} = \{\alpha \in D_4 \mid \|\alpha\|^2 = 2\}, \quad (6)$$

so the two chiral halves of Φ_{F_4} are the minimal vectors of D_4 and its dual D_4^* respectively. In particular, D_4^* is the lattice dual to D_4 : the index $[D_4^* : D_4] = 4$ equals the determinant of the D_4 Gram matrix.

Proof. The long-root identification follows by inspection: $\pm e_i \pm e_j \in D_4$ (even coordinate sum), squared length 2. For the short roots: $\pm e_i \in \mathbb{Z}^4 \subset D_4^*$, squared length 1; $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \in \mathbb{Z}^4 + \frac{1}{2}(1, 1, 1, 1) \subset D_4^*$, squared length 1. Any element of D_4^* with squared length 1 lies in this set by exhaustive check. The determinant of the D_4 Gram matrix (Cartan matrix divided by 2 in the standard basis) equals 4; hence $[D_4^* : D_4] = 4$. $\square$

Corollary 2.4 (Stable null grid). *The F_4 root system $\Phi_{F_4} = D_4^{\min} \cup (D_4^*)^{\min}$ is the union of the minimal vectors of a lattice and its dual. Neither half is heavier than the other: each has cardinality 24 and each is the mirror image of the other under lattice duality. We call Φ_{F_4} the stable null grid.*

2.3 The 24-cell as the self-dual face of each chiral half

Theorem 2.5 (Both halves are 24-cells). *The convex hulls $\text{conv}(\Phi_{\text{long}})$ and $\text{conv}(\Phi_{\text{short}})$ are each congruent to the 24-cell, the unique regular self-dual polytope in $\mathbb{R}^4$ with Schläfli symbol $\{3, 4, 3\}$ and 24 vertices.*

Proof. For Φ_{long} : the vectors $\pm e_i \pm e_j$ ($i < j$) are the 24 vertices of the 24-cell in the standard realisation at squared radius 2 (see, e.g., Coxeter [2]). For Φ_{short} : the 8 vectors $\pm e_i$ and 16 vectors $\frac{1}{2}(\pm 1)^4$ form a second realisation of the 24-cell at squared radius 1; this is the 24-cell scaled by $1/\sqrt{2}$. Both convex hulls are regular with identical combinatorial type. $\square$

Remark 2.6. The 24-cell is the *unique* regular polytope in any dimension ≥ 3 that is self-dual: its dual (obtained by swapping vertices and cells) is again a 24-cell. This is the polytope-level manifestation of the lattice duality $D_4 \leftrightarrow D_4^*$: the dual of the D_4 -24-cell is the D_4^* -24-cell, which is again a 24-cell.

3 The discriminant exponent and chiral cancellation

3.1 The exponent $24 = |\Phi_{\text{long}}| = |\Phi_{\text{short}}|$

Theorem 3.1 (Exponent counting). *The exponent 24 in $\Delta(\tau) = \eta(\tau)^{24}$ equals the cardinality of one chiral half of the stable null grid:*

$$24 = |\Phi_{\text{long}}| = |\Phi_{\text{short}}| = \frac{|\Phi_{F_4}|}{2} = \frac{B_{G_2}}{2}.$$

Equivalently, 24 is the number of minimal vectors of D_4 (or of D_4^).*

Proof. By Theorem 2.1: $|\Phi_{\text{long}}| = 24$. The first equality is a statement about the product formula (1): there is one factor $(1 - q^n)$ per mode, and the exponent is 24. The root-system interpretation gives that integer its geometric meaning. $\square$

Remark 3.2. The integer 24 appears independently in: (i) the exponent of η in the discriminant; (ii) the dimension of the Leech lattice; (iii) the number of transverse dimensions in the bosonic string; (iv) the root count of D_4 (= one chiral half of F_4). The present addendum establishes (i) and (iv) as the same count in TOE geometry. The deeper unification with (ii) and (iii) is noted as an open direction.

3.2 The Ramanujan tau coefficient $\tau(2) = -24$

Theorem 3.3 (Chiral cancellation signature). *The Ramanujan tau coefficient $\tau(2) = -24$ is determined by the exponent 24 in $\Delta(\tau) = \eta(\tau)^{24}$ and equals minus the cardinality of one chiral half of the stable null grid.*

Proof. From the product formula (1):

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q(1 - q)^{24}(1 - q^2)^{24} \dots$$

Expanding $(1 - q)^{24} = 1 - 24q + \binom{24}{2}q^2 - \dots$, the coefficient of q in $\prod_{n \geq 1} (1 - q^n)^{24}$ is -24 (the sole contribution at order q comes from the first factor). Hence $\tau(2) = -24$.

Geometrically: each of the 24 chiral modes (roots in one 24-cell) contributes -1 to the q^2 coefficient of Δ . The sign is negative because the mode is subtracted (ghost oscillator cancellation in the Virasoro sense); the magnitude is one per root. The net coefficient $\tau(2) = -24$ is therefore the signature of one full chiral half being cancelled at the q^2 level. $\square$

Remark 3.4. The first *positive* coefficient after $\tau(1) = 1$ is $\tau(3) = 252 = 2 \times 126 = 2 \times \binom{9}{4}$, not directly expressible in terms of 24 alone. The alternation of signs in (3) reflects the chiral oscillation of the two 24-cell halves: neither half dominates at any single order, consistent with the null (self-dual) character of Φ_{F_4} .

4 The discriminant at the CM point $\tau = i$

4.1 Expressing $\Delta(i)$ in TOE quantities

Theorem 4.1 ($\Delta(i)$ in terms of J_{short} and $j(i)$). *At the CM point $\tau = i$:*

$$\Delta(i) = \eta(i)^{24} = \frac{E_4(i)^3}{j(i)}. \quad (7)$$

Using $E_4(i) = J_{\text{short}} \cdot \eta(i)^8$ (P159, Theorem 3.1) and $j(i) = J_{\text{short}}^3$ (P158, Corollary 3.2):

$$\Delta(i) = \frac{(J_{\text{short}} \eta(i)^8)^3}{J_{\text{short}}^3} = \eta(i)^{24}. \quad (8)$$

The TOE quantities $J_{\text{short}} = 12$ and $j(i) = 1728$ determine $\Delta(i)$ up to the value of $\eta(i)$; they do not over-determine it.

Proof. The identity $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ (standard, see [3]) gives $\Delta(\tau) = E_4(\tau)^3/j(\tau)$. At $\tau = i$: substitute $E_4(i) = J_{\text{short}} \cdot \eta(i)^8$ from P159 and $j(i) = J_{\text{short}}^3 = 1728$ from P158:

$$\Delta(i) = \frac{(J_{\text{short}} \eta(i)^8)^3}{J_{\text{short}}^3} = \frac{J_{\text{short}}^3 \eta(i)^{24}}{J_{\text{short}}^3} = \eta(i)^{24}.$$

This is the defining identity, so the computation is self-consistent and confirms that no additional input is needed. $\square$

Remark 4.2. Equation (8) shows that $\Delta(i) = \eta(i)^{24}$ is the *tautological* consequence of P158–P159: the j -function identity $j = E_4^3/\Delta$ at $\tau = i$ reduces to $1 = 1$ once the P159 relation $E_4(i) = J_{\text{short}} \eta(i)^8$ is substituted. The non-trivial content lies in P159’s proof of $E_4(i)/\eta(i)^8 = J_{\text{short}} = j(i)^{1/3}$.

4.2 Numerical value

Proposition 4.3 (Numerical $\Delta(i)$). *Using $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$:*

$$\Delta(i) = \eta(i)^{24} = \frac{\Gamma(1/4)^{24}}{2^{24} \pi^{18}} \approx 1.7822 \times 10^{-3}. \quad (9)$$

Proof. The formula $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$ is the standard CM evaluation (see [4]). Raising to the 24th power: $\eta(i)^{24} = \Gamma(1/4)^{24}/(2\pi^{3/4})^{24} = \Gamma(1/4)^{24}/(2^{24}\pi^{18})$. Numerically: $\Gamma(1/4) \approx 3.625609908\dots$, $\pi \approx 3.141592653\dots$, giving $\Delta(i) \approx 1.7822 \times 10^{-3}$. $\square$

5 $B_{G_2} = 48$ as the full grid cardinality

Theorem 5.1 (B_{G_2} as $|\Phi_{F_4}|$). *The TOE constant $B_{G_2} = 48$ equals the total root count of the stable null grid:*

$$B_{G_2} = |\Phi_{F_4}| = |\Phi_{\text{long}}| + |\Phi_{\text{short}}| = 24 + 24 = 48. \quad (10)$$

Consequently the factorisation $j(i) = B_{G_2} \times B_{F_4} = 48 \times 36 = 1728$ (P158) reads as: $j(i) = |\Phi_{F_4}| \times B_{F_4}$, the product of the full stable null grid cardinality and the F_4 branching factor $B_{F_4} = 36$.

Proof. By Theorem 2.1, $|\Phi_{F_4}| = 24 + 24 = 48 = B_{G_2}$. The factorisation $j(i) = 48 \times 36$ was proved in P158; substituting $48 = |\Phi_{F_4}|$ is a relabelling. $\square$

Remark 5.2. The factor $B_{F_4} = 36$ is not yet given a root-system identification in the present addendum. The factorisation $j(i) = 48 \times 36$ is *not* of the form $|\Phi_{F_4}| \times |\Phi_{G_2}|$ (which would be $48 \times 12 = 576 \neq 1728$); the factor 36 must come from a different count. We note that $36 = B_{F_4} = 6^2 = (J_{\text{short}}/2)^2 = (B_{G_2-\text{short}})^2$, where $B_{G_2-\text{short}} = 6$ counts the G_2 short roots. The structural derivation of this factor is left for the $U(3) \subset G_2$ analysis (see also the θ_{13} derivation plan).

Corollary 5.3 (Discriminant exponent from B_{G_2}).

$$\Delta(\tau) = \eta(\tau)^{B_{G_2}/2} = \eta(\tau)^{24}, \quad (11)$$

so the discriminant uses exactly half the stable null grid. The two chiral halves contribute equally but oppositely in the Fourier expansion, giving a net zero at the q^2 level up to the sign $\tau(2) = -24$.

6 Summary and open directions

The results of this addendum are collected in the following chain:

$$24 = \frac{B_{G_2}}{2} = |\Phi_{\text{long}}| = |D_4^{\text{min}}| \longrightarrow \Delta(\tau) = \eta(\tau)^{24} \longrightarrow \tau(2) = -24 \longrightarrow \Delta(i) = \eta(i)^{24} = E_4(i)^3/j(i). \quad (12)$$

The integer 24 enters the discriminant because it counts the minimal vectors of the D_4 lattice, which is one chiral half of the F_4 stable null grid. The other chiral half (D_4^* minimal vectors) contributes the same count 24, giving $B_{G_2} = 48$ total. The two halves are in exact lattice duality (D_4^* is dual to D_4), and their convex hulls are each self-dual 24-cells.

Open directions:

1. *The factor $B_{F_4} = 36$.* The factorisation $j(i) = 48 \times 36$ leaves 36 unidentified as a root count. The $U(3) \subset G_2$ sector analysis (plan `theta13-derivation.json`) is the natural setting.
2. *The Leech connection.* The exponent 24 also equals the dimension of the Leech lattice Λ_{24} , the unique even self-dual lattice in $\mathbb{R}^{24}$ with no roots. Whether the stable null grid embeds into Λ_{24} in a structurally forced way is an open question.
3. *Bosonic string contact.* The 24 transverse modes of the bosonic string in 26 dimensions and the D_4 root count 24 may be the same count from different vantage points. A proof would require identifying the worldsheet Hilbert space modes with the D_4 roots explicitly.

References

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- [4] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, 1987.