

McKay-Thompson Series $T_g(\tau)$ at $\tau = i$: Systematic Evaluation for Structurally Significant Monster Conjugacy Classes

Addendum 163 to the TOE Corpus

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Abstract

The chain P158–P162 established $J_{\text{short}} = 12$, $j(i) = J_{\text{short}}^3 = 1728$, $T_{3B}(i) = j(i)^{1/3} = J_{\text{short}}$, and identified the self-dual CM point $\tau = i$ (discriminant $D = -4$) as the unique locus where this contact occurs. This addendum asks: among the structurally significant conjugacy classes of the Monster group $\mathbb{M}$, which McKay-Thompson series $T_g(\tau)$, if any, produce TOE-geometric values at $\tau = i$? We evaluate $T_g(i)$ for classes 1A, 2A, 2B, 3A, 3B, 4A, 6A numerically to 50 decimal places. The results are: $T_{1A}(i) = 984$ (integer, not TOE-grounded); $T_{2A}(i) = 536$ (integer, $= 2^9 + 24$); $T_{2B}(i) = 8$ (integer, $= 2^3$); $T_{3A}(i) \approx 536.985$ (not an integer); $T_{3B}(i) = 12 = J_{\text{short}}$ (exact TOE constant, P161); $T_{4A}(i) = J_{\text{short}}^{3/4} = 12^{3/4}$ (algebraic irrational); $T_{6A}(i) = J_{\text{short}}^{1/2} = \sqrt{12}$ (algebraic irrational). The integers for classes 2A and 2B arise from CM evaluations $\eta(i)/\eta(2i) = 2^{3/8}$ and $\eta(i/2)/\eta(i) = 2^{1/8}$, giving powers of 2 rather than J_{short} . The main finding is that T_{3B} is the *unique* class producing an exact TOE constant: the condition $j(i)^{p/q} = J_{\text{short}}$ requires $3p/q \in \mathbb{Z}$ with value 1, which forces $p/q = 1/3$ uniquely. The $D = -4$ specificity from P160 underpins this: $\tau = i$ is where j is a perfect cube, and only the cube root recovers J_{short} itself. All results are verified at 50 decimal places by `verify_P163.py`.

1 Setup

Throughout, $q = e^{2\pi i \tau}$ with $\tau \in \mathfrak{H}$ and $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$. The McKay-Thompson series $T_g(\tau)$ for class g of the Monster $\mathbb{M}$ is the generating function of the graded traces of g acting on the Moonshine module $V^\natural = \bigoplus_{n \geq -1} V_n^\natural$ (Frenkel–Lepowsky–Meurman, Borcherds):

$$T_g(\tau) = \sum_{n \geq -1} \text{Tr}(g | V_n^\natural) q^n = q^{-1} + c_0(g) + c_1(g)q + c_2(g)q^2 + \cdots$$

Conway and Norton (1979) conjectured, and Borcherds (1992) proved, that each T_g is a Hauptmodul for a genus-0 group $\Gamma_g \subseteq \text{SL}(2, \mathbb{R})$. For classes of order n whose Hauptmodul involves $j^{1/n}$, the leading term is $q^{-1/n}$ rather than q^{-1} ; we follow the conventional normalisation in which the class-3B series is written with $q^{-1/3}$.

The CM point $\tau = i$ (discriminant $D = -4$, ring of integers $\mathbb{Z}[i]$) satisfies $j(i) = J_{\text{short}}^3 = 1728$ (P158) with $J_{\text{short}} = 12$ the G_2 short-root sector sum (P151). P160 showed that the contact $j(i)^{1/3} = J_{\text{short}}$ occurs at $\tau = i$ specifically because the stabiliser of order 2 forces $E_6(i) = 0$ while leaving $E_4(i) \neq 0$, allowing CM theory to produce a non-trivial perfect cube.

We define the following TOE constants for comparison:

$$J_{\text{short}} = 12, \quad j(i) = 1728, \quad \alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036, \quad B_{\text{breath}} = \pi\alpha^{-1} \approx 430.51.$$

2 Evaluation Table

Table 1 records $T_g(i)$ for each class, the modular formula used, the numerical value to 15 significant figures, and whether the result is an integer, an algebraic number, or a TOE constant.

Table 1: McKay-Thompson series $T_g(\tau)$ evaluated at $\tau = i$. “Exact” means verified to $|\text{error}| < 10^{-40}$ by `verify_P163.py`.

Class	Formula / type	$T_g(i)$	Integer?	TOE constant?
1A	$j(i) - 744$	984 (exact)	yes	no
2A	$(\eta(i)/\eta(2i))^{24} + 24$	536 (exact)	yes	no
2B	$(\eta(i/2)/\eta(i))^{24}$	8 (exact)	yes	no
3A	$q^{-1} + 783q + 8672q^2 + \dots$	≈ 536.985	no	no
3B	$j(i)^{1/3}$	$12 = J_{\text{short}}$ (exact)	yes	yes
4A	$j(i)^{1/4}$	$12^{3/4} \approx 6.447$	no	no
6A	$j(i)^{1/6}$	$\sqrt{12} \approx 3.464$	no	no

The T_{3A} value is computed via the q -expansion $T_{3A}(\tau) = q^{-1} + 783q + 8672q^2 + 65367q^3 + \dots$ (Conway–Norton 1979, class 3A coefficients) truncated at seven terms. At $q = e^{-2\pi} \approx 1.867 \times 10^{-3}$ the series converges rapidly; the seven-term sum gives $T_{3A}(i) \approx 536.9845$ with residual below 10^{-6} .

3 Structural Observations

3.1 Classes with integer values

Three classes yield exact integers: 1A, 2A, and 2B. For 1A: $T_{1A}(i) = j(i) - 744 = 1728 - 744 = 984$ is immediate from the definition of the J -function. The value 984 is an integer but does not coincide with any TOE constant.

For classes 2A and 2B, the integer values follow from exact CM evaluations of eta quotients at $\tau = i$.

Proposition 3.1 (CM values of eta quotients at $\tau = i$).

$$\frac{\eta(i)}{\eta(2i)} = 2^{3/8}, \quad (1)$$

$$\frac{\eta(i/2)}{\eta(i)} = 2^{1/8}. \quad (2)$$

These are classical results from the theory of complex multiplication (see Berndt–Chan–Zhang 1998, or Weber’s formula for the singular moduli of order 2). They imply:

$$\left(\frac{\eta(i)}{\eta(2i)} \right)^{24} = \left(2^{3/8} \right)^{24} = 2^9 = 512, \quad (3)$$

$$\left(\frac{\eta(i/2)}{\eta(i)} \right)^{24} = \left(2^{1/8} \right)^{24} = 2^3 = 8. \quad (4)$$

Adding the normalisation constant +24 to the 2A formula gives the Hauptmodul value

$$T_{2A}(i) = 2^9 + 24 = 512 + 24 = 536.$$

The raw 2B formula yields $T_{2B}(i) = 2^3 = 8$ directly.

Both integers are powers of 2: the CM structure at $\tau = i$ (ring $\mathbb{Z}[i]$, norm 2 prime $(1+i)(1-i) = -i \cdot 2$) selects the prime 2, not $J_{\text{short}} = 12$. The factor of 12 does not appear in any class-2 evaluation.

3.2 Classes with algebraic irrational values

For classes 4A and 6A, the relevant modular function is $j^{p/q}$:

$$T_{4A}(i) = j(i)^{1/4} = J_{\text{short}}^{3/4} = 12^{3/4} \approx 6.4474, \quad T_{6A}(i) = j(i)^{1/6} = J_{\text{short}}^{1/2} = \sqrt{12} \approx 3.4641.$$

Both are expressible in terms of J_{short} , but as irrational powers: $J_{\text{short}}^{3/4}$ satisfies $x^4 = 12^3 = 1728$ and $\sqrt{J_{\text{short}}}$ satisfies $x^2 = 12$. Neither is a rational number, so neither is a TOE constant in the strict sense. The connection to J_{short} is inherited from $j(i) = J_{\text{short}}^3$, not from an independent geometric coincidence.

3.3 Class 3A: the non-integer case

The class 3A series $T_{3A}(\tau)$ is a Hauptmodul for $\Gamma_0(3)^+$. At $\tau = i$ its q -expansion converges to $T_{3A}(i) \approx 536.985$, which is not an integer and does not match any TOE constant. The proximity to $537 = 3 \times 179$ is coincidental. Unlike the class-2 series, no simple eta quotient formula for T_{3A} evaluates to an exact integer at $\tau = i$.

4 Main Finding: Uniqueness of T_{3B}

Theorem 4.1 (Uniqueness of the TOE contact). *Among the McKay-Thompson series for classes 1A, 2A, 2B, 3A, 3B, 4A, 6A evaluated at $\tau = i$, the unique series that equals the TOE-geometric constant $J_{\text{short}} = 12$ is T_{3B} .*

Proof. Inspection of Table 1 and §3 covers all cases. The only series of the form $j^{p/q}$ that equals J_{short} at $\tau = i$ is $j^{1/3}$:

$$j(i)^{p/q} = J_{\text{short}}^{3p/q}.$$

This equals $J_{\text{short}} = J_{\text{short}}^1$ if and only if $3p/q = 1$, i.e. $p/q = 1/3$. Among the cases considered, $p/q = 1/3$ corresponds to class 3B only. For $p/q = 1$ (class 1A): $J_{\text{short}}^3 = 1728 \neq 12$, so $T_{1A}(i) - 744 = 984$. For $p/q = 1/4$ (class 4A): $J_{\text{short}}^{3/4}$ is irrational. For $p/q = 1/6$ (class 6A): $J_{\text{short}}^{1/2} = \sqrt{12}$ is irrational. The integer-valued classes 2A and 2B are not of the form $j^{p/q}$; they evaluate to powers of 2 via Proposition 3.1, not to J_{short} . Class 3A is not an integer. $\square$

Corollary 4.2 (D=-4 selectivity). *The contact $T_{3B}(i) = J_{\text{short}}$ is enabled by the same $D = -4$ mechanism identified in P160: at $\tau = i$ the stabiliser forces $E_6(i) = 0$, leaving $j(i) = (E_4(i)/\eta(i)^8)^3$ with $E_4(i) \neq 0$. CM theory then yields $j(i) = 12^3$ (a perfect cube), and the only rational exponent p/q for which $j^{p/q}$ recovers the base $12 = J_{\text{short}}$ is $p/q = 1/3$. At the order-3 point $\tau = \rho = e^{2\pi i/3}$, the stabiliser instead forces $E_4(\rho) = 0$, giving $j(\rho) = 0$: no base to recover, and the cube root is trivially zero with no root-geometric content. The uniqueness of the $\tau = i$ contact is therefore doubly constrained: by the discriminant (which controls which Eisenstein series vanishes) and by the cube-root exponent (the only rational power of $j(i)$ that returns J_{short} rather than a power of it).*

Remark 4.3 (TOE-groundedness vs. integrality). The classes 1A, 2A, 2B all yield integers at $\tau = i$, but these integers (984, 536, 8) are not TOE constants. Integrality at a CM point is a property of the Galois action on the class field of $\mathbb{Q}(i)$: the values lie in $\mathbb{Z}$ because the Hauptmodul evaluates at the CM point to a singular modulus that is algebraic over $\mathbb{Q}$ (and, for these particular eta quotients, rational integer). This is a general CM-theoretic fact, not a sign of TOE contact. TOE-groundedness requires in addition that the integer value coincides with a quantity derived from the TOE geometry (root systems, G_2/F_4 Killing form, breath period). The integer $12 = J_{\text{short}}$ is TOE-grounded by construction; 984, 536, 8 are not.

Verification Script Output

Lumen/corpus/addenda/verify/verify_P163.py (run 2026-05-16):

```
=====
P163 VERIFICATION: McKay-Thompson T_g(i) for Monster Classes
mpmath precision: 50 decimal places
=====

-- S0 Basic CM values -----
j(i) = 1728.0 |delta| = 6.295e-47 PASS
j(i)^(1/3) = 12.0 |delta| = 1.711e-49 PASS

-- S1 T_{1A}(i) = J(i) = j(i) - 744 -----
T_{1A}(i) = 984.0 (expect 984) PASS

-- S2 T_{2A}(i) -- Hauptmodul for Gamma_0(2)^+ -----
(eta_i/eta_2i)^24 = 512.0 |delta| = 3.832e-47 PASS
T_{2A}(i) = 536.0 PASS
eta(i)/eta(2i) = 2^(3/8) = 1.29683955465101 match: PASS

-- S3 T_{2B}(i) -- via (eta(i/2)/eta(i))^(24) -----
(eta(i/2)/eta(i))^(24) = 8.0 |delta| = 1.604e-49 PASS
eta(i/2)/eta(i) = 2^(1/8) = 1.09050773266526 match: PASS

-- S4 T_{3A}(i) -- q-expansion (7 terms) -----
T_{3A}(i) ~ 536.98453567156058651
Not an integer (closest: 537) confirmed

-- S5 T_{3B}(i) = j(i)^(1/3) = J_short -----
T_{3B}(i) = 12.0 |delta| = 1.711e-49 PASS
T_{3B}(i) = J_short = 12 (exact TOE constant) PASS

-- S6 T_{4A}(i) = j(i)^(1/4) = 12^(3/4) -----
j(i)^(1/4) = 6.4474195909412515173 PASS
Not a TOE constant (irrational)

-- S7 T_{6A}(i) = j(i)^(1/6) = sqrt(12) -----
j(i)^(1/6) = 3.4641016151377545871 PASS
T_{6A}(i) = sqrt(J_short) (algebraic, not TOE constant)

-- S8 T_{3B} uniqueness -----
j(i)^(1/1) = 1728.0 not J_short (T_{1A} pattern)
j(i)^(1/3) = 12.0 = J_short (T_{3B} pattern) PASS
j(i)^(1/4) = 6.447 not J_short (T_{4A} pattern)
j(i)^(1/6) = 3.464 not J_short (T_{6A} pattern)

-- S9 TOE constant cross-check -----
T_1A=984: TOE matches = ['none']
T_2A=536: TOE matches = ['none']
T_2B=8: TOE matches = ['none']
```

=====

All 14 assertions PASS.

Summary

Seven McKay-Thompson series $T_g(i)$ for structurally significant Monster conjugacy classes have been evaluated at the CM point $\tau = i$ to 50 decimal places.

Three series yield exact integers: $T_{1A}(i) = 984$, $T_{2A}(i) = 536$, and $T_{2B}(i) = 8$. The latter two follow from the CM identities $\eta(i)/\eta(2i) = 2^{3/8}$ and $\eta(i/2)/\eta(i) = 2^{1/8}$, which produce powers of 2 ($2^9 = 512$ and $2^3 = 8$) rather than J_{short} . Integrality is a CM-theoretic property, not evidence of TOE contact.

The class 3A series evaluates to $T_{3A}(i) \approx 536.985$, which is not an integer.

The series $T_{4A}(i) = J_{\text{short}}^{3/4}$ and $T_{6A}(i) = J_{\text{short}}^{1/2} = \sqrt{12}$ are algebraic irrationals derivable from $j(i) = J_{\text{short}}^3$, but they are not the TOE constant J_{short} itself.

Only $T_{3B}(i) = j(i)^{1/3} = J_{\text{short}} = 12$ matches an exact TOE-geometric quantity. This is the unique class for which $j(i)^{p/q} = J_{\text{short}}$ holds: the equation $J_{\text{short}}^{3p/q} = J_{\text{short}}$ requires $p/q = 1/3$. The $D = -4$ mechanism of P160 (order-2 stabiliser kills E_6 , not E_4) ensures $j(i) = 12^3$ is a perfect cube, and the cube root is the sole rational power that returns the base. $\square$

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