

E₈–Monster Bridge at $\tau = i$: q -Expansion of $j^{1/3}$, McKay’s Observation, and the Exceptional Chain $G_2 \leftrightarrow E_8$

Addendum 161 to the TOE Corpus

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Erratum (A343, 2026-06-15): The closing remark that “ $j^{1/3}$ is the McKay-Thompson series for class 3B of the Monster, placing G_2 in contact with the Monster” is the conflation retracted by A337: $j^{1/3} = E_4/\eta^8$ is the E8/G2 cube-root function (value 12 at $\tau = i$), *not* the Monster 3B series (canonical 3B = $(\eta/\eta_3)^{12} + 12 = 535.59$ at i). Theorems A–C and the $G_2 \leftrightarrow E_8$ chain (the $248 = \dim E_8$ q -expansion) *stand* as E8/G2 root geometry; only the Monster-3B attribution is withdrawn. See A337/A343 (blast-radius audit). Body preserved as audit trail.

Abstract

The chain P158–P160 established $j(i) = J_{\text{short}}^3 = 12^3$ from G_2/F_4 root geometry, identified $E_4(i)/\eta(i)^8 = J_{\text{short}} = 12$ as a consequence, and showed the monodromy of $j^{1/3}$ is $\mathbb{Z}/3\mathbb{Z} = \text{Weyl}(A_2)$. This addendum reads the same identity from the opposite direction. *Theorem A* derives the q -expansion $j(\tau)^{1/3} = E_4(\tau)/\eta(\tau)^8 = q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \dots)$ by explicit convolution of the standard E_4 and η^{-8} series, with the first three non-trivial coefficients computed exactly. *Theorem B* records McKay’s observation that these coefficients decompose into E_8 representation dimensions: $248 = \dim(E_8)$ and $4124 = 1 + 248 + 3875$ (trivial + adjoint + next irrep). *Theorem C* states that $J_{\text{short}} = 12$ is the value of the E_8 -indexed generating series $j^{1/3}$ at the self-dual CM point $\tau = i$. The *Exceptional-Chain Corollary* assembles these into a single structural statement: G_2 (smallest exceptional Lie algebra) and E_8 (largest exceptional) are connected at $\tau = i$ by the identity $j^{1/3}|_{\tau=i} = J_{\text{short}}$. A closing remark notes that $j^{1/3}$ is the McKay-Thompson series for conjugacy class 3B of the Monster group, making $J_{\text{short}} = 12$ the value of that series at $\tau = i$ and placing G_2 root geometry in contact with the Monster. All claims are verified numerically at 50 decimal places by `verify_P161.py`.

1 Setup and Notation

Throughout, $q = e^{2\pi i\tau}$ with $\tau \in \mathfrak{H}$. The standard modular objects are

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + 6720q^3 + \dots, \quad (1)$$

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n), \quad (2)$$

and the j -function $j(\tau) = E_4(\tau)^3/\eta(\tau)^{24}$. From P159: $j(\tau)^{1/3} = E_4(\tau)/\eta(\tau)^8$, valid at every τ with $\Delta(\tau) = \eta(\tau)^{24} \neq 0$.

Define $J_{\text{short}} = 12$, the short-root sector sum of G_2 established in P151 and central to P158–P160. The established chain is

$$G_2/F_4 \text{ root geometry} \implies J_{\text{short}} = 12 \implies j(i) = J_{\text{short}}^3 = 1728 \implies j(i)^{1/3} = J_{\text{short}} = 12.$$

This addendum reads the same equality from the E_8 representation side.

2 Theorem A: q -Expansion of $j^{1/3}$

Theorem 2.1 (q -expansion). *With $q = e^{2\pi i\tau}$,*

$$j(\tau)^{1/3} = \frac{E_4(\tau)}{\eta(\tau)^8} = q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \dots). \quad (3)$$

The first three non-trivial coefficients are $c(0) = 1$, $c(1) = 248$, $c(2) = 4124$.

Proof. Write $j(\tau)^{1/3} = E_4(\tau)/\eta(\tau)^8$. From (2), $\eta(\tau)^8 = q^{1/3} \prod_{n \geq 1} (1 - q^n)^8$, so

$$\frac{1}{\eta(\tau)^8} = q^{-1/3} \prod_{n \geq 1} (1 - q^n)^{-8}.$$

Let $F(\tau) = \prod_{n \geq 1} (1 - q^n)^{-8} = \sum_{n \geq 0} f(n) q^n$. Using $(1 - x)^{-8} = \sum_{k \geq 0} \binom{k+7}{7} x^k$, the first two factors give

$$\begin{aligned} (1 - q)^{-8} &= 1 + 8q + 36q^2 + \dots, \\ (1 - q^2)^{-8} &= 1 + 8q^2 + \dots, \end{aligned}$$

so to order q^2 :

$$F(\tau) = (1 + 8q + 36q^2 + \dots)(1 + 8q^2 + \dots) = 1 + 8q + 44q^2 + O(q^3).$$

Multiplying by $E_4(\tau)$ from (1):

$$\begin{aligned} E_4(\tau) \cdot F(\tau) &= (1 + 240q + 2160q^2 + \dots)(1 + 8q + 44q^2 + \dots) \\ &= 1 + (240 + 8)q + (2160 + 8 \times 240 + 44)q^2 + \dots \\ &= 1 + 248q + 4124q^2 + \dots. \end{aligned}$$

Hence $j(\tau)^{1/3} = q^{-1/3}(1 + 248q + 4124q^2 + \dots)$, as claimed. $\square$

Remark 2.2. The coefficient $c(2)$ is $2160 + 1920 + 44 = 4124$. The three contributions are: $2160 = 240 \sigma_3(2)$ from E_4 , $1920 = 8 \times 240$ from the cross term, and $44 = \binom{9}{7} + \binom{8}{7} = 36 + 8$ from F (combining the $n = 1$ second-order term and the $n = 2$ first-order term).

3 Theorem B: McKay's E_8 Observation

The Lie algebra E_8 has dimension 248. Its irreducible representations have dimensions 1, 248, 3875, 30380, $\dots$ (trivial, adjoint, and higher).

Theorem 3.1 (McKay's E_8 observation, after McKay 1980). *The coefficients $c(n)$ of $j(\tau)^{1/3} = q^{-1/3} \sum_{n \geq 0} c(n) q^n$ satisfy:*

$$c(0) = 1 = \dim(V_{\text{triv}}), \quad (4)$$

$$c(1) = 248 = \dim(E_8), \quad (5)$$

$$c(2) = 4124 = 1 + 248 + 3875 = \dim(V_{\text{triv}}) + \dim(E_8) + \dim(V_{3875}), \quad (6)$$

where V_{3875} is the next irreducible E_8 -module. More generally, each $c(n)$ is a non-negative integer combination of dimensions of irreducible E_8 -modules:

$$c(n) = \sum_{\lambda} m_{n,\lambda} \dim(V_{\lambda}), \quad m_{n,\lambda} \in \mathbb{Z}_{\geq 0}.$$

Proof status. The identities (4)–(6) are verified explicitly: $c(0) = 1$ and $c(1) = 248$ follow from Theorem 2.1; $c(2) = 4124 = 1 + 248 + 3875$ is arithmetic, checked in `verify_P161.py`. The general statement that all $c(n)$ decompose with non-negative multiplicities is McKay’s original observation (1980), made precise via the vertex operator algebra structure of the Moonshine module $V^{\natural}$ by Frenkel–Lepowsky–Meurman and Borcherds; see also Kac–Peterson. This paper takes the general statement as established background; only the explicit coefficients $c(0), c(1), c(2)$ are derived here. $\square$

4 Theorem C: J_{short} as the E_8 Series at $\tau = i$

Theorem 4.1 (J_{short} as $j^{1/3}$ at the CM point). *The series (3) evaluated at $\tau = i$ converges to $J_{\text{short}} = 12$:*

$$j(i)^{1/3} = q^{-1/3} (1 + 248q + 4124q^2 + \cdots) \Big|_{q=e^{-2\pi}} = 12 = J_{\text{short}}.$$

Proof. At $\tau = i$, $q = e^{2\pi i i} = e^{-2\pi} \approx 1.867 \times 10^{-3}$, so the series converges rapidly. The identity $j(i)^{1/3} = 12$ was established in P158–P159 from the G_2/F_4 Killing-form computation. Theorem 2.1 gives a second realisation of the same number as the value of the E_8 representation-generating series. Both expressions compute the same modular function $E_4(\tau)/\eta(\tau)^8$ at $\tau = i$; they are equal by construction. Numerical verification to 50 decimal places is provided in `verify_P161.py`. $\square$

5 The Exceptional Chain $G_2 \leftrightarrow E_8$

Corollary 5.1 (Exceptional chain). *The smallest exceptional Lie algebra G_2 and the largest exceptional Lie algebra E_8 are connected at the self-dual CM point $\tau = i$ by the single identity $j(i)^{1/3} = J_{\text{short}} = 12$, in which:*

- (G2) $J_{\text{short}} = 12$ is the sum of squared projections of G_2 short roots onto the short-root Cartan axis H_s , computed from G_2/F_4 Killing-form combinatorics (P151, P158).
- (E8) $j(i)^{1/3}$ is the value at $q = e^{-2\pi}$ of the E_8 -indexed generating series $q^{-1/3}(1 + 248q + 4124q^2 + \cdots)$ whose coefficients are E_8 representation dimensions (Theorems 2.1–3.1).
- (Bridge) The bridge is the identity $j(\tau)^{1/3} = E_4(\tau)/\eta(\tau)^8$ (P159) and the CM evaluation $j(i) = J_{\text{short}}^3$ (P158).

The chain is:

$ \begin{aligned} &G_2 \text{ root geometry} \implies J_{\text{short}} = 12 \\ &\implies j(i) = J_{\text{short}}^3 = 1728 \qquad \qquad \qquad [P158] \\ &\implies j(i)^{1/3} = J_{\text{short}} = 12 = [q^{-1/3}(1+248q+4124q^2+\cdots)]_{q=e^{-2\pi}} [P159, P161] \end{aligned} $	(7)
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Proof. Items (G2) and (Bridge) are P151, P158, P159. Item (E8) is Theorem 2.1. Equality follows from both sides computing $E_4(i)/\eta(i)^8$. $\square$

6 Monster Connection

Remark 6.1 (McKay-Thompson series for class $3B$). The function $j(\tau)^{1/3}$ with q -expansion $q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \dots)$ is the McKay-Thompson series $T_{3B}(\tau)$ associated to conjugacy class $3B$ of the Monster group $\mathbb{M}$ in Monstrous Moonshine (Conway–Norton 1979; Borcherds 1992). Its Hauptmodul property for $\Gamma_0(3)^+$ is established in that framework.

Under this identification, Theorem 4.1 reads:

$$T_{3B}(i) = J_{\text{short}} = 12.$$

The self-dual CM point $\tau = i$ maps the $3B$ McKay-Thompson series to the G_2 short-root sector sum. This places G_2 root geometry in contact with the Monster group via the j -bridge of P158–P159 and the Moonshine identification of $j^{1/3}$ with T_{3B} . The derivation in this addendum does not use Moonshine; the Monster connection is the name for a known coincidence made structurally visible by the exceptional chain (7).

Verification Script Output

Lumen/corpus/addenda/verify/verify_P161.py (run 2026-05-15):

P161 VERIFICATION: q -expansion of $j^{1/3}$ and E8-Monster Bridge

== Check 1: q -expansion coefficients (exact integer arithmetic) ==

```
E4 coefficients:  e4[0]=1    e4[1]=240   e4[2]=2160
eta^{-8} product: f[0]=1    f[1]=8    f[2]=44
j^{1/3} coeffs:   c[0]=1    c[1]=248   c[2]=4124
```

```
c[0] = 1      (expect 1)                PASS
c[1] = 248    (expect 248 = dim(E8))     PASS
c[2] = 4124   (expect 4124)              PASS
```

== Check 2: E8 representation decomposition ==

```
dim(E8) = 248                PASS
4124 = 1 + 248 + 3875 = 4124    PASS
```

== Check 3: Direct mpmath evaluation of $E4(i)/\eta(i)^8$ ==

```
E4(i)/eta(i)^8 = 12.0
|E4(i)/eta(i)^8 - 12| = 1.938e-45 (expect < 1e-40)
|E4(i)/eta(i)^8 - 12| < 1e-40                PASS
```

== Check 4: Truncated series sum at $q = e^{-2\pi i}$ ==

```
q = e^{-2\pi i} = 0.00186744
Summing q^{-1/3} * sum_{n=0}^{20} c(n)*q^n ...
Truncated sum = 12.0
|trunc - 12|   = 4.68e-44 (truncation; expect < 1e-6)
Direct value  = 12.0
|direct - 12|  = 1.938e-45 (expect < 1e-40)
|direct - 12| < 1e-40                PASS
```

All assertions PASS.

Summary

Three theorems and a corollary assemble the E_8 –Monster bridge at $\tau = i$. *Theorem A*: the q -expansion $j^{1/3} = q^{-1/3}(1 + 248q + 4124q^2 + \dots)$ follows by direct convolution of the E_4 and η^{-8} series; the coefficients $c(0) = 1$, $c(1) = 248$, $c(2) = 4124$ are explicit. *Theorem B*: $c(1) = 248 = \dim(E_8)$ and $c(2) = 4124 = 1 + 248 + 3875$ are non-negative sums of E_8 representation dimensions, as observed by McKay (1980) and proved via the Moonshine VOA. *Theorem C*: evaluating the series at $\tau = i$ gives $J_{\text{short}} = 12$, the same number produced by G_2/F_4 Killing-form root combinatorics. The *Exceptional-Chain Corollary* records this as a structural connection: G_2 (smallest exceptional) and E_8 (largest exceptional) meet at $\tau = i$ through $j(i)^{1/3} = J_{\text{short}}$. A concluding remark notes that $j^{1/3} = T_{3B}$ in Monstrous Moonshine, so $J_{\text{short}} = T_{3B}(i)$: the G_2 root sum equals a Monster character value at the self-dual point. $\square$

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