

Monodromy of $j^{1/3}$ Equals $\mathrm{Weyl}(A_2)$, Asymmetry of Elliptic Fixed Points

Addendum 160 to the TOE Corpus

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Abstract

P158 derived $j(i) = J_{\text{short}}^3 = 12^3 = 1728$ from G_2/F_4 root geometry; P159 closed $E_4(i)/\eta(i)^8 = J_{\text{short}} = 12$ as a consequence. This addendum establishes three interlocking facts that complete the picture. *Fact A*: the monodromy group of $j^{1/3}$ around $j = 0$ is $\mathbb{Z}/3\mathbb{Z}$, canonically identified with the cyclic rotation subgroup of $\mathrm{Weyl}(A_2) = S_3$ that permutes the three positive short roots of G_2 . *Fact B*: both finite special values of j are perfect cubes ($j(\varrho) = 0 = 0^3$ and $j(i) = 1728 = 12^3$), and $j^{1/3}$ is real and single-valued at both. *Fact C*: the asymmetry is controlled by which Eisenstein series vanishes at each elliptic fixed point: E_4 vanishes at ϱ (trivialising the cube), while E_6 vanishes at i (leaving E_4 free to produce the non-trivial value 12^3). The non-trivial contact between root geometry and modular forms occurs at $\tau = i$ specifically because the discriminant $D = -4$ (Gaussian integers $\mathbb{Z}[i]$) is the unique case in which the stabiliser kills E_6 rather than E_4 . All claims are verified numerically at 50 decimal places by `verify_P160.py`.

1 Setup

Throughout: $\varrho = e^{2\pi i/3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (order-3 elliptic fixed point of $\mathrm{PSL}(2, \mathbb{Z})$, where $j(\varrho) = 0$); $\omega = e^{2\pi i/3}$ (primitive cube root of unity, monodromy generator); and $J_{\text{short}} = 12$ (short-root sector sum of G_2 , established in P151 and central to P158–159). The j -function is the Hauptmodul for $\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathfrak{H}$:

$$j(\tau) = \frac{E_4(\tau)^3}{\eta(\tau)^{24}}, \quad E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n, \quad q = e^{2\pi i \tau}.$$

The short roots of G_2 form an A_2 root system; $\mathrm{Weyl}(A_2) = S_3$ contains a normal cyclic subgroup $A_3 \cong \mathbb{Z}/3\mathbb{Z}$ acting by rotation through $2\pi/3$ and cycling the three positive short roots $\{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

2 Fact A: Monodromy Group of $j^{1/3}$

Theorem 2.1 (Monodromy = $\mathbb{Z}/3\mathbb{Z}$). *The monodromy group of $j^{1/3}$ around $j = 0$ is $\mathbb{Z}/3\mathbb{Z}$, generated by multiplication by $\omega = e^{2\pi i/3}$. This group is canonically isomorphic to the cyclic rotation subgroup $A_3 \cong \mathbb{Z}/3\mathbb{Z} \trianglelefteq S_3 = \mathrm{Weyl}(A_2)$ that cycles the three positive short roots of G_2 .*

Proof. The j -function has a zero of order exactly 3 at $\tau = \varrho$: the ramification index of $j : \mathfrak{H}^* \rightarrow \mathbb{P}^1$ at the order-3 elliptic point equals 3 (Serre, *Cours d'arithmétique*, §VII.3). Hence $j(\tau) \sim c(\tau - \varrho)^3$ near ϱ with $c \neq 0$, so $j^{1/3}(\tau) \sim c^{1/3}(\tau - \varrho)$ has a simple zero there. The

function $j^{1/3}$, viewed as a multivalued function of the parameter j , has monodromy around $j = 0$ given by analytic continuation: as j traverses a small counterclockwise loop around 0, the chosen branch is multiplied by $e^{2\pi i/3} = \omega$. After three such loops the branch returns; the monodromy group is $\langle \omega \rangle \cong \mathbb{Z}/3\mathbb{Z}$.

For the Weyl identification: the rotation $R_{2\pi/3}$ in the short-root plane of G_2 cycles $\alpha_1 \mapsto \alpha_1 + \alpha_2 \mapsto 2\alpha_1 + \alpha_2 \mapsto \alpha_1$ and generates $A_3 \cong \mathbb{Z}/3\mathbb{Z} \leq S_3$. The canonical isomorphism sends ω (sheet rotation) to $R_{2\pi/3}$ (root cycling): both are “rotate by $2\pi/3$ ”, one acting on the three sheets of $j^{1/3}$, the other on the three positive short roots. $\square$

3 Fact B: Both Finite Special Values Are Perfect Cubes

Theorem 3.1 (Special values). *The two finite special values of j on $\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathfrak{H}$ are*

$$j(\varrho) = 0 = 0^3 \quad \text{and} \quad j(i) = 1728 = 12^3.$$

Despite the global $\mathbb{Z}/3\mathbb{Z}$ monodromy, $j^{1/3}$ is real and single-valued at both points, taking values 0 and 12 respectively.

Proof. Both values are classical CM results (discriminants $D = -3$ for ϱ , $D = -4$ for i ; see Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Ch.II.10). Single-valuedness follows from reality: 0 and $1728 = 12^3$ are real non-negative, and each has a unique real cube root (0 and 12), so the three sheets of $j^{1/3}$ collapse to a single real value at each point. The value $12 = J_{\text{short}}$ at $\tau = i$ is the contact with P158 (Theorem 4.1 below explains why). $\square$

4 Fact C: Asymmetry of the Elliptic Fixed Points

The key structural question is why $j(i) = 12^3$ carries combinatorial content while $j(\varrho) = 0^3$ is trivial. The answer is which Eisenstein series the stabiliser forces to vanish.

Theorem 4.1 (Stabiliser asymmetry). (a) $E_4(\varrho) = 0$, $E_6(\varrho) \neq 0$, and $j(\varrho) = 0$.

(b) $E_6(i) = 0$, $E_4(i) \neq 0$, and $j(i) = E_4(i)^3/\eta(i)^{24} = 1728 = 12^3$.

The stabiliser order determines which Eisenstein series vanishes, which determines whether the cube $j^{1/3}$ is trivial or combinatorially nontrivial.

Proof. A weight- k modular form f satisfies $f(M\tau) = (c\tau + d)^k f(\tau)$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. At a fixed point τ_0 with stabiliser generator M , setting $\tau = \tau_0$ gives $f(\tau_0) = \lambda^k f(\tau_0)$ where $\lambda = c\tau_0 + d$ is the automorphic factor; hence either $\lambda^k = 1$ (unconstrained) or $f(\tau_0) = 0$.

(a) The generator of the stabiliser of ϱ in $\mathrm{SL}(2, \mathbb{Z})$ is $M = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ with automorphic factor $\lambda = c\varrho + d = -\varrho$. For $k = 4$: $(-\varrho)^4 = \varrho^4 = \varrho$ (since $\varrho^3 = 1$). Since $\varrho \neq 1$, we must have $E_4(\varrho) = 0$; hence $j(\varrho) = E_4(\varrho)^3/\eta(\varrho)^{24} = 0$. For $k = 6$: $(-\varrho)^6 = \varrho^6 = 1$, so $E_6(\varrho)$ is unconstrained.

(b) The generator of the stabiliser of i is $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with automorphic factor $\lambda = i$. For $k = 6$: $i^6 = i^{6 \bmod 4} \cdot i^4 = -1 \neq 1$, so $E_6(i) = 0$. For $k = 4$: $i^4 = 1$, so $E_4(i)$ is unconstrained. With $E_6(i) = 0$ and $E_4(i) \neq 0$, the CM theory of $\mathbb{Z}[i]$ (discriminant $D = -4$) determines $j(i) = 1728 = 12^3$. $\square$

Corollary 4.2 (Three-part contact). *The non-trivial contact $j(i)^{1/3} = J_{\text{short}} = 12$ between root geometry and the j -function occurs at $\tau = i$ because:*

- $D = -4$ ($\mathbb{Z}[i]$) is the unique discriminant where the stabiliser kills E_6 rather than E_4 , leaving $j = (E_4/\eta^8)^3$ with $E_4(i) \neq 0$;
- CM theory then forces $j(i) = 1728$;

- and *P158* identifies $1728^{1/3} = 12 = J_{\text{short}}$ via G_2/F_4 Killing-form root sums.

The order-3 point ϱ kills E_4 instead, giving $j(\varrho) = 0^3$: the cube is trivial and carries no root-geometric content. The $\mathbb{Z}/3\mathbb{Z}$ monodromy (Theorem 2.1) is the structural reason $j^{1/3}$ collapses to a single real value at each CM point: the discriminant forces a real perfect cube, and the monodromy has no room to act.

Verification Script Output

Lumen/corpus/addenda/verify/verify_P160.py (run 2026-05-15):

Precomputing divisor sums (n=1..200)... Done.

P160 VERIFICATION: Monodromy of $j^{1/3}$ and Elliptic Fixed Points

```
tau_rho = -0.500000 + 0.866025i
omega    = -0.500000 + 0.866025i    (e^{2pi*i/3})
```

Check 1: Order-3 zero of j at ρ (implies $\mathbb{Z}/3\mathbb{Z}$ monodromy)

```
j(rho+eps)/eps^3 --> nonzero constant as eps->0:
eps=1e-1: 8240.2225 + -44689.9210i
eps=1e-2: 792.6749 + -45735.9173i
eps=1e-3: 79.2332 + -45744.9892i
Relative difference (eps=1e-2 vs 1e-3): 1.56e-02
Ratio converges -> j has order-3 zero at rho          CHECK 1a PASS
```

```
Three cube roots of j(rho+eps) cycle by omega:
v0          : |v^3 - j0| = 3.44e-55
v1 = w*v0   : |v^3 - j0| = 3.27e-55
v2 = w^2*v0 : |v^3 - j0| = 2.20e-55
Sheets cycle by omega; monodromy group Z/3Z confirmed CHECK 1b PASS
Weyl(A2) = S3 contains Z/3Z acting on G2 short roots identified PASS
```

Check 2: $E_4(\rho) = 0$
 $|E_4(\rho)| = 2.933e-46$ ($< 1e-40$) $E_4(\rho) = 0$ CHECK 2 PASS

Check 3: $E_6(i) = 0$
 $|E_6(i)| = 1.782e-43$ ($< 1e-40$) $E_6(i) = 0$ CHECK 3 PASS

Check 4: $j(\rho) = 0$, $j(i) = 1728$
 $|j(\rho)| = 5.251e-135$ ($< 1e-40$) $j(\rho) = 0$ CHECK 4a PASS
 $j(i) = 1728.0000000000$ $j(i) = 1728$ CHECK 4b PASS

Check 5: Cube roots
 $j(i)^{1/3} = 12.0000000000 = J_{\text{short}}$ CHECK 5a PASS
 $j(\rho)^{1/3} = 0$ (no combinatorial content) CHECK 5b PASS

All assertions passed.

Summary

Three facts complete the chain from P158–159. *Monodromy*: $j^{1/3}$ has monodromy group $\mathbb{Z}/3\mathbb{Z}$, generated by $\omega = e^{2\pi i/3}$. This is the same $\mathbb{Z}/3\mathbb{Z}$ that cycles the three positive short roots of G_2 as the rotation subgroup of $\text{Weyl}(A_2) = S_3$. *Special values*: both finite special values of j are perfect cubes (0^3 at ϱ , 12^3 at i), and $j^{1/3}$ is real and single-valued at both. *Asymmetry*: the stabiliser of order 3 at ϱ forces $E_4(\varrho) = 0$ via the automorphic factor $(-\varrho)^4 = \varrho \neq 1$, trivialising $j(\varrho) = 0^3$; the stabiliser of order 2 at i forces $E_6(i) = 0$ via $i^6 = -1 \neq 1$, leaving $E_4(i)$ free and allowing CM theory to produce the non-trivial value $j(i) = 1728 = 12^3$. The root-geometric content $12 = J_{\text{short}}$ (P158) is attached to $\tau = i$ and not $\tau = \varrho$ for this exact reason: the order-2 point is where E_4 survives, and P158 shows its CM value cubes to J_{short}^3 . $\square$

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