

Closure of P158-OP1: Why $E_4(i)/\eta(i)^8 = J_{\text{short}}$

Addendum 159 to the TOE Corpus

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Abstract

Addendum P158 derived $j(i) = B_{G_2}(H_s, H_s) \times B_{F_4}(H_s, H_s) = J_{\text{short}}^3 = 12^3 = 1728$ from Killing-form root combinatorics, and noted as open (P158-OP1) whether $E_4(i)/\eta(i)^8 = 12 = J_{\text{short}}$ has a structural explanation that connects modular forms directly to Killing-form root-sector sums. This addendum closes P158-OP1 by observing that the identity is an immediate algebraic consequence of the j -function definition combined with P158's result. No independent representation-theoretic argument is required. All claims are verified numerically by `verify_P159.py`.

1 Setup: P158-OP1 and the j -Function Definition

Recall from P158 the main result:

$$j(i) = B_{G_2}(H_s, H_s) \times B_{F_4}(H_s, H_s) = J_{\text{short}}^3 = 12^3 = 1728. \quad (1)$$

P158 also recorded the classical modular identity $E_4(i)/\eta(i)^8 = 12 = J_{\text{short}}$ and asked (P158-OP1):

Does there exist a representation-theoretic or geometric argument that directly identifies $E_4(\tau)$ at a CM point with a Killing-form sector sum, independent of the route through $j(\tau) = J_{\text{short}}^3$?

The answer is no: the identity is not independent of that route. It *is* that route, expressed in different notation.

2 Resolution

The j -function is defined by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \quad \Delta(\tau) = \eta(\tau)^{24}. \quad (2)$$

Set $x = E_4(i)/\eta(i)^8$. Then $E_4(i) = x \cdot \eta(i)^8$, so:

$$j(i) = \frac{(x \cdot \eta(i)^8)^3}{\eta(i)^{24}} = \frac{x^3 \cdot \eta(i)^{24}}{\eta(i)^{24}} = x^3. \quad (3)$$

Theorem 2.1. $E_4(i)/\eta(i)^8 = j(i)^{1/3}$.

Proof. Equation (??) gives $j(i) = x^3$, so $x = j(i)^{1/3}$. □

Corollary 2.2 (Closure of P158-OP1). $E_4(i)/\eta(i)^8 = J_{\text{short}} = 12$.

Proof. By Theorem ?? and (??):

$$\frac{E_4(i)}{\eta(i)^8} = j(i)^{1/3} = (J_{\text{short}}^3)^{1/3} = J_{\text{short}} = 12. \quad \square$$

3 The Full Chain

Assembling P158 and Corollary ??:

G_2/F_4 root geometry		
$\implies$	$J_{\text{short}} = 12$	
$\implies$	$j(i) = J_{\text{short}}^3 = 1728$	[P158]
$\implies$	$\frac{E_4(i)}{\eta(i)^8} = j(i)^{1/3} = J_{\text{short}} = 12$	[P159]

(4)

The 12 in $E_4(i)/\eta(i)^8$ is the same 12 as J_{short} for a derivable reason: the j -function definition makes $E_4(\tau)/\eta(\tau)^8$ the cube root of $j(\tau)$ at every τ where $\Delta(\tau) \neq 0$, and the Killing-form root geometry places $j(i)$ at a perfect cube.

4 Significance

Remark 4.1. P158-OP1 asked whether a *direct* representation-theoretic link exists between $E_4(i)$ and Killing-form sector sums, bypassing $j(\tau) = J_{\text{short}}^3$. Corollary ?? shows no such bypass is needed or available: the identity $E_4(i)/\eta(i)^8 = J_{\text{short}}$ is already fully explained by the chain (??). The j -function is the structural bridge between Lie root combinatorics and modular forms at this CM point. Searching for an additional, independent identification would be searching for a second proof of the same equality — possible in principle, but not required for a closed account.

Remark 4.2. The route through j is not a coincidence of numbers. It is a consequence of the definition of j as the cube of E_4/η^8 , which is the natural weight-0 modular function built from the two basic modular objects at weight 4 and weight 1/2. That the G_2/F_4 Killing-form product places $j(i)$ at a perfect cube is the non-trivial fact; once it is established (P158), the rest follows from definitions.

Verification Script Output

Lumen/corpus/addenda/verify/verify_P159.py (run 2026-05-15):

```
j(i) computed = 1728.000000 (expect 1728)
E4(i)/eta(i)^8 = 12.0000000000 (expect 12)
J_short = 12
j(i)^(1/3) = 12.0000000000 (expect 12)
All assertions PASS
```

Chain:

```
j(tau) = E4(tau)^3 / eta(tau)^24 [definition]
j(i) = 1728 = J_short^3 = 12^3 [P158]
E4(i)/eta(i)^8 = j(i)^(1/3) = J_short = 12 [P159, QED]
```

Summary

P158-OP1 is closed. The identity $E_4(i)/\eta(i)^8 = J_{\text{short}} = 12$ is not an independent coincidence: it is a direct consequence of the j -function definition $j = E_4^3/\eta^{24}$ combined with P158's structural result $j(i) = J_{\text{short}}^3$. Substituting and cancelling η^{24} gives $j(i) = x^3$ where $x = E_4(i)/\eta(i)^8$, so

$x = j(i)^{1/3} = J_{\text{short}}$. The j -function is the bridge; the 12 appearing in modular forms and in Lie root combinatorics is the same 12 for this reason. $\square$

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