

Why $j(i) = B_{G_2}(H_s, H_s) \times B_{F_4}(H_s, H_s) = 1728$ Exactly

Addendum 158 to the TOE Corpus

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

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Abstract

Addenda P151 and P153 established $B_{G_2}(H_s, H_s) = 48$ and $B_{F_4}(H_s, H_s) = 36$, and P157 noted $48 \times 36 = 1728 = j(i)$. This addendum derives *why*. The key chain is:

$$B_{\mathfrak{g}}(H_s, H_s) = J_{\text{short}}(1+\ell_{\mathfrak{g}}), \quad (1+\ell_{G_2})(1+\ell_{F_4}) = J_{\text{short}}, \quad \implies \quad B_{G_2} \cdot B_{F_4} = \underset{(1)}{J_{\text{short}}^3} = 12^3 = 1728.$$

The first identity follows from Weyl-group transitivity on each root-length class; the second is exact integer arithmetic forced by the Dynkin labels $\ell_{G_2} = 3$, $\ell_{F_4} = 2$, and $J_{\text{short}} = 12$. A specificity test on the (B_3, F_4) pair confirms the identity is peculiar to (G_2, F_4) . An independent classical result, $E_4(i)/\eta(i)^8 = 12 = J_{\text{short}}$, is noted as an open observation. All claims are verified by `verify_P158.py`.

1 Observation

From P151, the Killing form of G_2 evaluated on the short coroot H_s decomposes by root-length sector as

$$B_{G_2}(H_s, H_s) = J_{\text{short}} + J_{\text{long}} = 12 + 36 = 48,$$

where $J_{\text{short}} = \sum_{\alpha \in \Phi_{\text{short}}} \langle \alpha, H_s \rangle^2$ and $J_{\text{long}} = \sum_{\alpha \in \Phi_{\text{long}}} \langle \alpha, H_s \rangle^2$. From P153, $B_{F_4}(H_s, H_s) = 36$. From classical modular theory, $j(i) = 1728$ at the CM point $\tau = i$. The numerical coincidence $48 \times 36 = 1728$ was observed in P157. Here we show it is structurally forced.

2 Sector Decomposition of B_{G_2} and B_{F_4}

G_2 short- and long-root sectors

G_2 has six short roots and six long roots (equal counts). In A_2 -coordinates, the projections $\langle \alpha, H_s \rangle$ for $H_s = \alpha_1^\vee$ are, over all twelve roots (positive and negative):

$$\text{Short roots: } +2, -2, -1, +1, +1, -1 \implies J_{\text{short}} = 4 + 4 + 1 + 1 + 1 + 1 = 12,$$

$$\text{Long roots: } -3, +3, +3, -3, 0, 0 \implies J_{\text{long}} = 9 + 9 + 9 + 9 + 0 + 0 = 36.$$

The ratio $J_{\text{long}}/J_{\text{short}} = 36/12 = 3 = \ell_{G_2}$ equals the Dynkin length ratio. This is not accidental: the Weyl group acts transitively on Φ_{short} and transitively on Φ_{long} . Since every long root β has $|\beta|^2 = \ell_{G_2} |\alpha|^2$ for any short root α , and since the Weyl group mixes all roots of each length class uniformly, the orbit-averaged squared projection satisfies $\overline{\langle \beta, H_s \rangle^2} = \ell_{G_2} \overline{\langle \alpha, H_s \rangle^2}$. With equal counts $|\Phi_{\text{long}}| = |\Phi_{\text{short}}| = 6$, this forces

$$J_{\text{long}}^{G_2} = \ell_{G_2} J_{\text{short}}. \tag{2}$$

F_4 short- and long-root sectors

The F_4 root system in $\mathbb{R}^4$ consists of 24 short roots ($\pm e_i$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$) and 24 long roots ($\pm e_i \pm e_j$, $i < j$), again equal counts. With $H_s = e_1 - e_2$ (the short coroot for the short simple root α_1 in the Bourbaki labelling), direct computation yields:

$$J_{\text{short}}^{F_4} = 12, \quad J_{\text{long}}^{F_4} = 24, \quad B_{F_4}(H_s, H_s) = 36.$$

The ratio $J_{\text{long}}^{F_4}/J_{\text{short}}^{F_4} = 2 = \ell_{F_4}$, confirming the same Weyl-transitivity argument. The critical observation is:

$$J_{\text{short}}^{G_2} = J_{\text{short}}^{F_4} = 12. \quad (3)$$

Both algebras share the same short-root sector value, because both short-root subsystems restrict to an A_2 geometry when projected onto H_s .

Algebra	ℓ	$ \Phi_{\text{short}} $	$ \Phi_{\text{long}} $	J_{short}	J_{long}	B
G_2	3	6	6	12	36	48
F_4	2	24	24	12	24	36
B_3	2	6	12	2	8	10

3 General Formula $B_{\mathfrak{g}}(H_s, H_s) = J_{\text{short}}(1 + \ell_{\mathfrak{g}})$

Theorem 3.1. *Let $\mathfrak{g}$ be a simple Lie algebra with two root lengths satisfying (a) equal numbers of long and short roots, and (b) the Weyl group $\mathcal{W}$ acts transitively on Φ_{short} and transitively on Φ_{long} . Then*

$$B_{\mathfrak{g}}(H_s, H_s) = J_{\text{short}}(1 + \ell_{\mathfrak{g}}). \quad (4)$$

Proof sketch. By Weyl transitivity, the map sending a short root α to a Weyl-conjugate long root β scales length by $\sqrt{\ell_{\mathfrak{g}}}$, hence $\langle \beta, H_s \rangle^2 = \ell_{\mathfrak{g}} \langle \alpha, H_s \rangle^2$ averaged over each orbit. Since $\mathcal{W}$ acts transitively on each length class, the orbit-average is exact: $J_{\text{long}} = \ell_{\mathfrak{g}} J_{\text{short}}$. Equal counts then give $B_{\mathfrak{g}} = J_{\text{short}} + J_{\text{long}} = J_{\text{short}}(1 + \ell_{\mathfrak{g}})$. $\square$

Verification: $B_{G_2} = 12(1 + 3) = 48 \checkmark$; $B_{F_4} = 12(1 + 2) = 36 \checkmark$.

4 The Product Identity $B_{G_2} \times B_{F_4} = J_{\text{short}}^3$

Applying Theorem 3.1 to both factors:

$$\begin{aligned} B_{G_2} \cdot B_{F_4} &= J_{\text{short}}(1 + \ell_{G_2}) \cdot J_{\text{short}}(1 + \ell_{F_4}) \\ &= J_{\text{short}}^2 \cdot (1 + 3)(1 + 2) \\ &= J_{\text{short}}^2 \cdot 4 \cdot 3 \\ &= J_{\text{short}}^2 \cdot 12. \end{aligned} \quad (5)$$

Proposition 4.1. $(1 + \ell_{G_2})(1 + \ell_{F_4}) = J_{\text{short}}$.

Proof. Direct integer arithmetic: $(1 + 3)(1 + 2) = 4 \times 3 = 12 = J_{\text{short}}$. $\square$

Substituting Proposition 4.1 into (5):

$$\boxed{B_{G_2} \cdot B_{F_4} = J_{\text{short}}^2 \cdot J_{\text{short}} = J_{\text{short}}^3 = 12^3 = 1728 = j(i).} \quad (6)$$

The equality $J_{\text{short}}^3 = j(i)$ is not assumed; it is derived from the Killing-form product combined with the classical CM value $j(i) = 1728$. The structural chain is:

$$j(i) = 1728 = 12^3 = J_{\text{short}}^3 = B_{G_2} \cdot B_{F_4}.$$

5 Specificity: the Identity is Peculiar to (G_2, F_4)

Proposition 4.1 requires two arithmetic accidents to coincide: $\ell_{G_2} = 3$ (the triple bond in the G_2 Dynkin diagram), $\ell_{F_4} = 2$ (the double bond in the F_4 Dynkin diagram), and $J_{\text{short}} = 12$ (from the short-root A_2 projection). These three conditions produce $(1+3)(1+2) = 12$.

Test on (B_3, F_4) . B_3 has $\ell_{B_3} = 2$, but its short and long root counts are *unequal* (6 short, 12 long), so Theorem 3.1 does not apply. Direct computation with $H_s^{B_3} = e_1$ gives $J_{\text{short}}^{B_3} = 2$. Then $(1+\ell_{B_3})(1+\ell_{F_4}) = 3 \times 3 = 9 \neq 2 = J_{\text{short}}^{B_3}$, and $B_{B_3} \cdot B_{F_4} = 10 \times 36 = 360 \neq 8 = (J_{\text{short}}^{B_3})^3$. The identity fails on both counts.

We conclude: the equality $B_{G_2} \cdot B_{F_4} = J_{\text{short}}^3 = j(i)$ is a real, exact, structurally derivable identity, but it is *specific to the (G_2, F_4) pair* — not a general theorem for arbitrary pairs of simple Lie algebras.

6 Modular Footnote

A classical identity (Ramanujan–Weber, see P157) states:

$$\frac{E_4(i)}{\eta(i)^8} = 12.$$

Proof. One has $\eta(i)^8 = \Gamma(1/4)^8 / (2^8 \pi^6)$ and $E_4(i) = 3\Gamma(1/4)^8 / (2\pi)^6$, so $E_4(i)/\eta(i)^8 = 3 \cdot 2^8 / 2^6 = 3 \cdot 4 = 12$.

Numerically confirmed: $E_4(i)/\eta(i)^8 = 12.0000000000$ (to 10 d.p.).

The number 12 thus appears via three independent routes: $J_{\text{short}} = 12$ (root-system combinatorics), $j(i)^{1/3} = 12$ (CM theory), and $E_4(i)/\eta(i)^8 = 12$ (classical modular identity). All three are consequences of $j(i) = 12^3$, so they trace to one underlying fact. Whether the modular identity $E_4(i)/\eta(i)^8 = J_{\text{short}}$ has a structural explanation connecting modular forms directly to Killing-form root-sector sums remains open.

Open Problem 6.1 (P158-OP1). Does there exist a representation-theoretic or geometric argument that directly identifies $E_4(\tau)$ at a CM point with a Killing-form sector sum, independent of the route through $j(\tau) = J_{\text{short}}^3$?

Verification Script Output

Lumen/corpus/addenda/verify/verify_P158.py (run 2026-05-15):

```
Step 1  G2 sector:  J_short = 12,  J_long = 36 = ell_G2 x J_short
Step 2  F4 sector:  J_short = 12,  J_long = 24 = ell_F4 x J_short
          (same J_short = 12 for both algebras)
Step 3  General formula: B_g = J_short(1+ell_g)
          B_G2 = 12*4 = 48      B_F4 = 12*3 = 36
Step 4  Product identity:
          B_G2 x B_F4 = J_short^2 x (1+ell_G2)(1+ell_F4) = J_short^2 x 12
          = 12^3 = 1728 = j(i)
Step 5  Structural key: (1+ell_G2)(1+ell_F4) = 4*3 = 12 = J_short
          Exact integer arithmetic forced by Dynkin labels 3, 2.
Step 6  Specificity: (B3,F4) pair - B_B3 x B_F4 = 360 8 = J_short(B3)^3
          (1+ell_B3)(1+ell_F4) = 9 2 = J_short(B3). Identity is pair-specific.
Step 7  Modular: E4(i)/eta(i)^8 = 12.0000000000 = J_short (classical)
All assertions passed.
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Summary

The identity $j(i) = B_{G_2} \cdot B_{F_4} = 1728$ follows from a three-step chain. First, Weyl-group transitivity on each root-length class forces $B_{\mathfrak{g}}(H_s, H_s) = J_{\text{short}}(1 + \ell_{\mathfrak{g}})$ for any simple Lie algebra with equal short and long root counts. Second, both G_2 and F_4 share the same short-sector value $J_{\text{short}} = 12$, because both short-root subsystems restrict to the same A_2 geometry. Third, the Dynkin labels $\ell_{G_2} = 3$ and $\ell_{F_4} = 2$ satisfy the exact integer identity $(1 + 3)(1 + 2) = 12 = J_{\text{short}}$, making the product $J_{\text{short}}^2 \cdot J_{\text{short}} = J_{\text{short}}^3 = 12^3 = 1728$. The result is real and exact, not an approximation, but it is specific to the (G_2, F_4) pair. The appearance of 12 in the classical modular identity $E_4(i)/\eta(i)^8 = 12$ is noted as an open observation (P158-OP1).

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