

Addendum P157

Modular Forms at CM Points: A Systematic Search for TOE Constant Structure

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1 Motivation

Addendum P153 established

$$B_{F_4}(H_s, H_s) \times B_{G_2}(H_s, H_s) = 36 \times 48 = 1728 = 12^3,$$

where B_{F_4} and B_{G_2} are the Killing forms of F_4 and G_2 evaluated on the distinguished element H_s of the principal $\mathfrak{sl}_2$. The observation was not anticipated; 1728 was derived, not sought.

Classical complex multiplication theory gives $j(i) = 1728$ at the CM point $\tau = i$ (discriminant $D = -4$, ring of integers $\mathbb{Z}[i]$). These two routes to the same integer are independent: one is Lie-theoretic, the other number-theoretic. The question for this addendum is whether this contact at $D = 4$ is the tip of a systematic structure — TOE constants appearing broadly in modular forms at CM points — or an isolated instance.

TOE constants under test (tolerance $< 0.5\%$):

$$\alpha^{-1} \approx 137.036, \quad \Omega = \pi\alpha^{-1} \approx 430.512, \quad \mathcal{P} = 432, \quad \mathcal{C} = 1728, \quad J_s = 12, \quad J_\ell = 36, \quad B_{G_2} = 48.$$

2 Systematic CM Table: $j(\tau_D)$ for $D = 3 \dots 163$

Let τ_D denote the CM point of fundamental discriminant $-D$. The table below lists the exact classical value $j(\tau_D)$, verifies it via q -expansion to 60 decimal places (using $j = E_4^3/\Delta$), and records any near-match ($< 0.5\%$) to a TOE constant.

D	$j(\tau_D)$ (exact)	TOE hit
3	0	—
4	1728	$ j = \mathcal{C}$; $ j ^{1/3} = J_s$; $ j /4 = \mathcal{P}$
7	-3375	—
8	8000	—
11	-32768	—
12	54000	—
16	287496	—
19	-884736	—
27	-12288000	—
28	16581375	—
43	-884736000	—
67	-147197952000	—
163	-262537412640768000	—

Verification output (abridged):

D	$j(\tau_D)$ exact	q-exp verified	TOE hits
3	0	OK	
4	1728	OK	$ j /4=\text{BREATH}$; $ j ^{(1/3)}=J_SHORT$; $ j /4=\text{PRODUCT}$; $ j =\text{CUBE}$
7	-3375	OK	
8	8000	OK	
11	-32768	OK	
12	54000	OK	
16	287496	OK	
19	-884736	OK	
27	-12288000	OK	
28	16581375	OK	
43	-884736000	OK	
67	-147197952000	OK	
163	-2.625e+17	OK	

$|j(\tau_{27})|^{(1/3)} = 230.759931$ (not a TOE constant)
 $|j(\tau_{43})|^{(1/3)} = 960.000000$ (= 960; $960/\alpha_{\text{inv}} = 7.005$)
 $|j(\tau_{163})|^{(1/3)} = 640320.000000$
 Only $D=4$ yields $j(i)=1728=12^3$ with $12=J_{\text{short}}$

Finding. $D = 4$ is the unique discriminant in the full Heegner set where $|j(\tau_D)|$ or $|j(\tau_D)|^{1/3}$ falls within 0.5% of a TOE constant. The cube root $|j(\tau_{43})|^{1/3} = 960$ is exact, but $960/\alpha^{-1} \approx 7.005$ — not a structurally meaningful ratio.

3 Eta and Eisenstein Series at $\tau = i$

3.1 Verification of known identities

At $\tau = i$, the q -expansion computations give (60 d.p.):

$\text{eta}(i)$ via q -product = 0.76822542232605661727
 $\text{eta}(i) = \Gamma(1/4)/(2 \pi^{3/4}) = 0.76822542232605661727$

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Match (60 dps):          True

E4(i)                    = 1.45576289226870936311
E6(i)                    = 1.03e-48  (-> 0 by symmetry of CM(i))
j(i) = E4^3/Delta        = 1728.000000  (exact: 1728)

E4(i) known 3*Gamma(1/4)^8/(2*pi)^6 = 1.45576289226870936311
Match:  True

```

The vanishing of $E_6(i)$ is classical: $\tau = i$ has automorphism of order 4 in $\mathrm{SL}_2(\mathbb{Z})$, forcing $E_6(i) = 0$.

3.2 Key finding: $E_4(i)/\eta(i)^8 = 12$ exactly

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*** E4(i) / eta(i)^8 = 12.0000000000 ***
This equals J_SHORT = 12 exactly.
Proof: eta(i)^8 = Gamma(1/4)^8 / (2^8 * pi^6)
      E4(i)      = 3 * Gamma(1/4)^8 / (2*pi)^6
      E4(i)/eta(i)^8 = 3/(2*pi)^6 * (2^8 * pi^6) = 3*4 = 12

```

Identity.

$$\frac{E_4(i)}{\eta(i)^8} = 12 = J_s.$$

This is a provable classical identity, derivable from the exact formulae $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$ and $E_4(i) = 3\Gamma(1/4)^8/(2\pi)^6$. The computation gives $J_s = 12$ by a third independent route: it already appears as J_{short} (P151) and as $j(i)^{1/3}$ (P153).

No other ratio of $E_4(i)$, $E_8(i) = E_4(i)^2$, or $E_{12}(i)$ to any power of π , $\Gamma(1/4)$, or α^{-1} falls within 0.5% of a TOE constant.

4 Ramanujan Near-Integer Search

The Heegner phenomenon: $e^{\pi\sqrt{163}} \approx 640320^3 + 744$ (Ramanujan, proved via $j(\tau_{163}) = -640320^3$). We search $D \in [1, 300]$ for

$$e^{\pi\sqrt{D}} \approx n \cdot \alpha^{-k}, \quad n \in \mathbb{Z}, \quad k \in \{-3, -2, -1, 1, 2, 3\},$$

with threshold $|\text{ratio} - n| < 0.01$.

Precision note. For $D \geq 43$ and $k = 3$, the ratio $e^{\pi\sqrt{D}}/\alpha^{-3}$ exceeds 10^{14} , exhausting `float64` fractional precision. All $D \geq 43$ hits with error ≈ 0 are precision artefacts; only $D \leq 40$ results are numerically meaningful.

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D <=40 hits (float64-reliable regime):
D= 2: e^{pi*sqrt(2)} ~ 218789488 x alpha^3,  error=0.009429
D= 5: e^{pi*sqrt(5)} ~ 21111036 x alpha^2,  error=0.005674
D=11: e^{pi*sqrt(11)} ~ 4591558 x alpha^1,  error=0.000505
D=21: e^{pi*sqrt(21)} ~ 245015497 x alpha^1, error=0.002836
D=29: e^{pi*sqrt(29)} ~ 1185 x alpha^{-2},  error=0.005190
D=30: e^{pi*sqrt(30)} ~ 216851 x alpha^{-1}, error=0.003012
D=33: e^{pi*sqrt(33)} ~ 3665 x alpha^{-2},  error=0.001578

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Hits in [1,40]: 7 (expected by chance: ~4.8)

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D = 43 (Heegner): e^{pi*sqrt(43)} = 884736743.9998
|j(tau_43)| + 744 = 884736744
e^{pi*sqrt(43)} / alpha_inv^3 = 343.801016 (not near integer)
```

Finding. Seven hits in [1, 40] against an expectation of ~ 4.8 from equidistribution. The slight excess is unsurprising at this sample size. None of the near-integers n has structural significance in the TOE. The Heegner discriminant $D = 43$ does not produce a near-integer ratio to α^{-k} for any small k .

5 Depth of the 1728 Connection

Exact integer arithmetic:

```
1728 = 12^3                True
1728 = B_F4 x B_G2 = 36x48 True
1728 = j(i)                classical CM theorem
1728 / 4 = 432 = PRODUCT   True
1728^(1/3) = 12 = J_SHORT   True
12 appears: J_SHORT, cbrt(j(i)), E4(i)/eta(i)^8 -- three independent routes
```

BREATH vs $j(i)/4$:

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j(i)/4 = 432.000000 (exact integer = J_short x J_long)
BREATH = pi * alpha_inv = 430.512245217399
j(i)/4 - BREATH = G1_abs = 1.487754782601 (P156, the known unknown)
relative gap    = G1_rel = 0.003455778086 (0.3456%)
```

The structural picture at $\tau = i$ is:

$$\begin{aligned}
j(i) &= 1728 = 12^3 = B_{F_4} \times B_{G_2} \\
j(i)^{1/3} &= 12 = J_s \\
j(i)/4 &= 432 = \mathcal{P} = J_s \times J_\ell \\
E_4(i)/\eta(i)^8 &= 12 = J_s \quad (\text{provable identity}) \\
E_6(i) &= 0 \quad (\text{CM symmetry at } \tau = i) \\
\Omega = \pi\alpha^{-1} &= 430.512\dots \neq 432 \quad (\text{gap} = G_1)
\end{aligned}$$

All roads through $\tau = i$ lead back to $12 = J_s$. The integer 432 is $j(i)/4$; the breath period Ω misses it by G_1 (the known unknown of P156, $\approx 0.35\%$). The chain is $j(i) \rightarrow 12 \rightarrow 432$ via exact arithmetic; Ω is not part of it.

6 Verdict

What was found.

1. $j(i) = 1728 = B_{F_4} \times B_{G_2}$: a genuine, exact coincidence. Both sides equal 1728 by independent computation.
2. $E_4(i)/\eta(i)^8 = 12 = J_s$: a classical, provable identity. The same 12 that is J_{short} in P151 and $j(i)^{1/3}$ in P153. One underlying arithmetic fact ($j(i) = 12^3$), three appearances of 12.
3. $j(i)/4 = 432 = \mathcal{P} = J_s \times J_\ell$: exact integers, no transcendental input. The breath period Ω differs by G_1 .

What was not found.

- No CM discriminant other than $D = 4$ yields a j -value within 0.5% of any TOE constant.
- No Eisenstein or eta combination at $\tau = i$ produces α^{-1} , Ω , ϕ , or $\sin^2 \theta_W$ within 0.5%.
- The Ramanujan scan ($D \leq 40$, reliable precision) produces hits statistically consistent with equidistribution; no structurally motivated near-integer emerges.

Conclusion. $j(i) = 1728$ is an isolated structural contact between CM theory and TOE root-system geometry. The contact is real and exact; it is not coincidence in the pejorative sense — both computations are forced. But it does not extend. The evidence does not support a hypothesis of systematic TOE constant structure in modular forms. The observation belongs with P153 as a boundary condition: something the correct unified theory must eventually account for, but not yet a derivation.

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[ged-traversal: query="modular forms CM points j-invariant alpha TOE constants" nodes="j-invariant  
CM point; Killing form root system; G1 known unknown; Eisenstein series" score=0.41]
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