

Addendum P155

SU(3)-Orbit Average of the G_2 Coroot Pairing and Gap G1

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Abstract

Addendum P154 exhausted six Laplace-type smearings of the discrete G_2 root spectral measure; none closes gap G1 without tautological construction. The one genuinely untried direction identified there is the *SU(3)-orbit average* of the squared coroot pairing: replacing each root's discrete contribution to the second moment by the Haar average of $B_{g_2}(\text{Ad}_g H_s, H_s)^2$ over $g \in \text{SU}(3) \subset G_2$. We compute this integral in two independent ways—analytic Schur orthogonality and a $N = 200\,000$ Monte Carlo over $\text{SU}(3)$ —and compare the result with the G_2 -Haar analogue and the target $\Omega/9$. The $\text{SU}(3)$ orbit average evaluates to the exact rational $18 = 12^2/8$, undershooting the target $\Omega/9 \approx 47.835$ by 62%, a discrepancy $180\times$ larger than G1. The G_2 Haar average gives $48^2/14 \approx 164.6$, overshooting by $3.4\times$. The discrete value 48 (P151) remains the unique approach to $\Omega/9$ within G1. A structural identity $48 = B_{\text{su}(3)}(H_s, H_s)^2/|\Phi^+(A_2)|$ is recorded. G1 is not closed.

1 Setup

Throughout, the TOE constants are

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \quad \Omega = \pi\alpha^{-1} \approx 430.512 \quad (\text{breath period}), \quad (1)$$

and the target second moment is

$$\mathcal{T} = \frac{\Omega}{9} \approx 47.835. \quad (2)$$

The discrete coroot Killing value from P151 is $B_{G_2}(H_s, H_s) = 48$, where H_s is the coroot of the short simple root of G_2 , normalised so that its projections onto the 12 G_2 roots produce the root integer $\{-3, -3, -2, -1, -1, 0, 0, 1, 1, 2, 3, 3\}$ with $\sum_{\alpha} \langle \alpha, H_s \rangle^2 = 48$. The gap is

$$\text{G1} = \frac{432 - \Omega}{\Omega} \approx 3.456 \times 10^{-3}. \quad (3)$$

The $\text{SU}(3)$ embedding is $A_2 \hookrightarrow G_2$ via the short-root subgroup. The Lie algebra decomposes as $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus W$ where $W \cong \mathbf{3} \oplus \bar{\mathbf{3}}$ as an $\mathfrak{su}(3)$ -module. H_s lies in the Cartan subalgebra of $\mathfrak{su}(3)$; we represent it as $H_s = \text{diag}(1, -1, 0) \in \mathfrak{su}(3) \subset \mathfrak{g}_2$.

2 The three integrals

Integral I: G_2 Haar average

By Schur orthogonality for the 14-dimensional real adjoint representation of G_2 :

$$I_{G_2} = \int_{G_2} B_{g_2}(\text{Ad}_g H_s, H_s)^2 dg = \frac{B_{G_2}(H_s, H_s)^2}{\dim G_2} = \frac{48^2}{14} = \frac{1152}{7} \approx 164.571. \quad (4)$$

This exceeds $\mathcal{T}$ by a factor of 3.44.

Integral II: SU(3) Haar average

Key formula. For $g \in \text{SU}(3)$, write $h = g H_s g^\dagger$ and let $h_1, h_2, h_3 = h_{11}, h_{22}, h_{33}$ be the (real) diagonal entries of h . The six roots of A_2 are $\pm(e_i - e_j)$ with projections

$$\alpha(H) = \langle e_i - e_j, H \rangle = H_{ii} - H_{jj}. \quad (5)$$

A direct root-sum calculation gives

$$B_{\text{su}(3)}(\text{Ad}_g H_s, H_s) = \sum_{\alpha \in \Phi(A_2)} \alpha(\text{Ad}_g H_s) \alpha(H_s) = 6(h_1 - h_2), \quad (6)$$

since $\alpha_1(H_s) = 2$, $\alpha_2(H_s) = -1$, $(\alpha_1 + \alpha_2)(H_s) = 1$ and the total contracts to $2(2h_1 - 2h_2 + h_1 - h_3 - h_2 + h_3) = 6(h_1 - h_2)$.

Schur result. Let $f(g) = B_{\text{su}(3)}(\text{Ad}_g H_s, H_s)$. By Schur orthogonality for the adjoint representation of $\text{SU}(3)$ (dimension $d = 8$, real orthogonal):

$$\int_{\text{SU}(3)} (\text{Ad}_g X)_i (\text{Ad}_g X)_j dg = \frac{B_{\text{su}(3)}(X, X)}{d} \delta_{ij}, \quad (7)$$

so

$$I_{\text{SU}(3)} = \int_{\text{SU}(3)} f(g)^2 dg = \frac{B_{\text{su}(3)}(H_s, H_s)^2}{d}. \quad (8)$$

The A_2 root projections onto H_s are $\{+2, -2, +1, -1, -1, +1\}$, giving $B_{\text{su}(3)}(H_s, H_s) = 4 + 4 + 1 + 1 + 1 + 1 = 12$, and therefore

$$I_{\text{SU}(3)} = \frac{12^2}{8} = \frac{144}{8} = 18. \quad (9)$$

Monte Carlo verification. We generate $N = 200\,000$ Haar-distributed $\text{SU}(3)$ matrices via QR decomposition of a complex Gaussian ensemble (Mezzadri 2007), compute $[6(h_1 - h_2)]^2$ for each sample, and average. The result 18.053 ± 0.048 agrees with (??) to within one standard error.

Integral III: flag-variety fibration

The flag variety G_2/T carries an $\text{SU}(3)$ -equivariant fibration $G_2/T \rightarrow G_2/\text{SU}(3)$ with fibre $\text{SU}(3)/T$. Under the Liouville measure, the fibre-average of the integrand $B_{g_2}(\text{Ad}_g H_s, H_s)^2$ over each fibre is exactly $I_{\text{SU}(3)} = 18$ by translation-invariance of the Haar measure. The base $G_2/\text{SU}(3)$ carries no additional modulation from H_s in the $\mathfrak{su}(3)$ direction. Thus the flag-variety second moment reduces to the $\text{SU}(3)$ Haar average computed above.

3 Script output (verbatim)

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=====
P155 FLAG-VARIETY ORBIT AVERAGE VERIFICATION
=====
ALPHA_INV          = 137.0363037759
BREATH_PERIOD      = 430.5122452174
TARGET (Omega/9)   = 47.8346939130
DISCRETE           = 48
G1 = (432-Omega)/Omega = 3.45577809e-03 (0.345578%)
```

```

--- INTEGRAL I: G2 HAAR AVERAGE (Schur, analytic) ---
I_G2 = B(H_s,H_s)^2 / dim(G2) = 48^2/14 = 164.571429
TARGET                                = 47.834694
ratio I_G2 / TARGET = 3.440420 (overshoots by 244.04%)

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--- INTEGRAL II: SU(3) HAAR AVERAGE (Schur, analytic) ---
A2 root projections onto H_s = diag(1,-1,0): [2,-2,1,-1,-1,1]
B_{su3}(H_s,H_s) = Sum alpha(H_s)^2 = 12
Schur formula: I_SU3 = B^2/dim(su3) = 12^2/8 = 18.000000
TARGET                                = 47.834694
ratio I_SU3 / TARGET = 0.376296 (undershoots by 62.37%)

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Structural identity: B_{su3}(H_s,H_s)^2/|Phi+(A2)| = 12^2/3 = 48.0
DISCRETE = 48 (match: True)

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--- INTEGRAL II: SU(3) HAAR AVERAGE (Monte Carlo, N=200,000) ---
I_SU3 (MC)          = 18.052622 +/- 0.047835 (1-sigma)
I_SU3 (analytic) = 18.000000
MC vs analytic discrepancy: 0.0526 (0.292%)
TARGET (Omega/9) = 47.834694
ratio I_SU3_MC / TARGET = 0.377396
gap I_SU3 - TARGET      = -29.7821

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SUMMARY TABLE
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Quantity		Value	Ratio/TARGET
Discrete B_{G2}(H_s,H_s)	[P151]	48.0000	1.003456
G2 Haar avg	48^2/14 [analytic]	164.5714	3.440420
SU(3) Haar avg	12^2/8 [analytic]	18.0000	0.376296
SU(3) Haar avg	[MC]	18.0526	0.377396
TARGET	Omega/9	47.8347	1.000000

```

--- INTERPOLATION ---
t* = (Omega/9 - 18) / (48 - 18) = 0.9944897971
1 - t* = 0.0055102029
(1-t*) as multiple of G1: 1.594490
(1-t*) = (432-Omega)/30 [no recognised geometric constant]

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=====
G1 CLOSING STATUS
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G1	= 0.345578%
I_SU3 - TARGET / TARGET	= 62.3704%
Ratio (orbit gap) / G1	= 180.48x

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CONCLUSION: I_SU3 = 18 undershoots TARGET = 47.8347 by 62.37%,
which is 180.5x G1. DISCRETE 48 remains the unique value within
G1 of Omega/9. G1 is NOT closed by orbit averaging.

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STRUCTURAL NOTE:

$$\begin{aligned} I_{\text{SU3}} \times \dim(\mathfrak{su}(3)) / |\Phi^+(A_2)| &= 18 \times 8/3 = 48.0 = \text{DISCRETE} \\ \text{DISCRETE} &= B_{\text{su3}}(H_s, H_s)^2 / |\Phi^+(A_2)| = 144/3 = 48 \end{aligned}$$

4 Findings

Summary of integral values

Quantity	Value	Ratio to $\mathcal{T} = \Omega/9$
Discrete $B_{G_2}(H_s, H_s)$ [P151]	48	1.003456
G_2 Haar average $48^2/14$	164.571	3.4404
SU(3) Haar average $12^2/8$ [analytic]	18	0.3763
SU(3) Haar average [MC]	18.053	0.3774
Target $\Omega/9$	47.835	1

The SU(3) orbit average is the exact rational 18. It undershoots the target by 62.37%, a gap that is $180.5\times$ the size of G1. The G_2 Haar average overshoots by $3.44\times$. The discrete value 48 is the only entry in the table that lies within G1 of the target.

Why the SU(3) orbit average does not close G1

Equation (??) gives $I_{\text{SU}(3)} = B^2/d$ where $B = 12$ is an integer (the A_2 Killing value) and $d = 8$ is an integer (the dimension of $\mathfrak{su}(3)$). The result 18 is therefore rational. The target $\mathcal{T} = \pi(4\pi^3 + \pi^2 + \pi)/9$ is transcendental. No rational equals a transcendental: the orbit average cannot close the gap regardless of the smearing strategy, because Schur collapses the entire Haar integral to an algebraic combination of the Killing norm and the representation dimension.

This is the same obstruction as in P154, now at the level of the full SU(3) Haar measure rather than a Laplace transform. Maximal smearing (Haar) does not help.

Structural identity

The computation reveals

$$48 = \frac{B_{\mathfrak{su}(3)}(H_s, H_s)^2}{|\Phi^+(A_2)|} = \frac{12^2}{3} = 48. \quad (10)$$

Equivalently, $I_{\text{SU}(3)} \times \dim(\mathfrak{su}(3))/|\Phi^+(A_2)| = 18 \times 8/3 = 48 = \text{DISCRETE}$. Equation (??) provides a third representation of the P151 discrete value: it equals the A_2 Schur second moment scaled by the dimension-to-positive-root ratio. This is an algebraic identity connecting the G_2 Killing norm to the SU(3) subgroup data; it does not introduce transcendental content.

Interpolation

The unique linear interpolation $I(t) = (1-t) I_{\text{SU}(3)} + t \cdot 48$ that hits $\mathcal{T}$ satisfies

$$t^* = \frac{\mathcal{T} - 18}{30} = \frac{\Omega/9 - 18}{30} \approx 0.99449. \quad (11)$$

The deviation from 1 is $1-t^* = (432-\Omega)/30 \approx 5.51 \times 10^{-3}$, a multiple of G1 with proportionality constant ≈ 1.59 . The denominator $30 = 48 - 18$ carries no recognised G_2 invariant. There is no natural geometric interpolation parameter with a recognisable value.

What remains

P154 (item 1 of its “what would be needed” list) called for a continuous measure on the G_2 root system whose second moment evaluates to $\Omega/9$ *from first principles*. The $SU(3)$ Haar measure is the most natural such candidate; it evaluates to 18, not $\Omega/9$. The direction is exhausted. Any resolution of G1 via spectral methods must originate from a structure that connects the G_2 root lattice to the S^3 holonomy geometry at the TOE scale — a layer of geometry not accessible through Schur orthogonality on any subgroup of G_2 .

Status of G1

G1 remains open. The $SU(3)$ -orbit average of $B_{g_2}(\text{Ad}_g H_s, H_s)^2$ equals the rational 18, falling short of $\Omega/9 \approx 47.835$ by 62% ($180.5 \times$ G1). The G_2 Haar average equals $48^2/14 \approx 164.6$, overshooting by $3.44 \times$. The discrete P151 value 48 remains the unique approach to $\mathcal{T}$ within G1. Schur orthogonality guarantees that any Haar integral over a compact subgroup of G_2 returns a rational multiple of Killing-form data; no such rational can equal the transcendental $\Omega/9$.

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