

Addendum P154

Laplace-Type Transform of the G_2 Root Spectral Measure and Gap G1

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Abstract

Gap G1 = $(432 - \pi\alpha^{-1})/(\pi\alpha^{-1}) \approx 0.3456\%$ remains open after P151–P153. The integer $432 = J_{\text{short}} \times J_{\text{long}}$ emerges from G_2 Killing norms, while the breath period $\Omega = \pi\alpha^{-1}$ arises from a continuous S^3 integral. We ask whether a Laplace-type smearing of the discrete G_2 root spectral measure $\mu = \sum_{\alpha} \delta(t - \langle \alpha, H_s \rangle)$ converts the integer second moment $\sum_{\alpha} \langle \alpha, H_s \rangle^2 = 48$ into the transcendental target $\Omega/9 \approx 47.835$. Six candidates (A–F) are computed numerically. All candidates that formally hit the target do so either by tautological construction or because G1 < 1% makes the discrete value already within the tolerance threshold. No candidate provides geometric input that explains the transcendental correction. G1 remains open.

1 Setup: G_2 root spectral measure

Fix the TOE constants

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \quad \Omega = \pi\alpha^{-1} \approx 430.512 \quad (\text{breath period}). \quad (1)$$

G_2 has 12 roots. The projections of all roots onto the coroot H_s of the short simple root α_1 are

$$\{\langle \alpha, H_s \rangle\} = \{-3, -3, -2, -1, -1, 0, 0, 1, 1, 2, 3, 3\}. \quad (2)$$

The *discrete root spectral measure* is

$$\mu_{\text{disc}} = \sum_{\alpha \in \Delta} \delta(t - \langle \alpha, H_s \rangle). \quad (3)$$

Its second moment is the Killing form restricted to H_s :

$$M_2^{\text{disc}} = \int t^2 d\mu_{\text{disc}}(t) = \sum_{\alpha} \langle \alpha, H_s \rangle^2 = 48. \quad (4)$$

The product of short and long Killing norms gives $J_s \times J_l = 12 \times 36 = 432$, recovering the integer approximant of Ω . The residual gap is

$$\text{G1} = \frac{432 - \Omega}{\Omega} = \frac{432 - \pi\alpha^{-1}}{\pi\alpha^{-1}} \approx 3.456 \times 10^{-3}. \quad (5)$$

The hypothesis under investigation: some natural smearing of μ_{disc} at the breath scale yields a continuous second moment equal to $\Omega/9 \approx 47.835$, closing G1. (The factor 9 appears because $M_2^{\text{disc}} = 48$ and $\Omega/9 \approx 48$, so $\Omega/9$ is the “corrected” first-moment-squared-over-roots comparison.)

2 Candidates A–F: definitions and numerical results

All computations are performed by `verify_P154.py` (see §??). A candidate **passes** if its result agrees with its declared target to within 1%; a pass due to tautological construction or to $G1 < 1\%$ is flagged explicitly.

Candidate A — Gaussian smearing at $\sigma^2 = \pi$. Convolving μ_{disc} with a Gaussian of variance $\sigma^2 = \pi$ shifts the second moment by σ^2 per root:

$$M_2^A = \sum_{\alpha} [\langle \alpha, H_s \rangle^2 + \pi] = 48 + 12\pi \approx 85.70. \quad (6)$$

This far exceeds $\Omega/9 \approx 47.84$. **FAIL.**

Candidate B — Laplace MGF, mean-square tilt. Define the one-sided Laplace functional

$$L(s) = \sum_{\alpha} e^{-s\langle \alpha, H_s \rangle^2} = 4e^{-9s} + 2e^{-4s} + 4e^{-s} + 2, \quad s \geq 0. \quad (7)$$

L has range $(2, 12]$; neither $\alpha^{-1} \approx 137$ nor $\alpha \approx 0.0073$ lies in this range, so there is no s^* with $L(s^*) = \alpha^{\pm 1}$. The *weighted mean-square* is $W(s) = -L'(s)/L(s)$, with $W(0) = 48/12 = 4$ and $W(\infty) \rightarrow 0$. We seek s^* such that $N \cdot W(s^*) = \Omega/9$, i.e.

$$-\frac{L'(s^*)}{L(s^*)} = \frac{\Omega}{108} \approx 3.9863. \quad (8)$$

Bisection gives $s^* \approx 9.847 \times 10^{-4}$. The leading-order approximation uses

$$\left. \frac{d}{ds} \left[-\frac{L'}{L} \right] \right|_{s=0} = \frac{-\sum p^4 \cdot N + (\sum p^2)^2}{N^2} = \frac{-360 \times 12 + 2304}{144} = -14, \quad (9)$$

so $s^* \approx (4 - \Omega/108)/14 = (432 - \Omega)/1512$. The factor $1512 = 14 \times 108$ carries no recognisable G_2 invariant; it encodes the fourth-moment $\sum p^4 = 360$ and the number of roots. Concretely, $s^* \times \Omega \approx 0.4239$ and $s^* \times 432 \approx 0.4254$, neither a simple constant. Candidate B **passes numerically** but with $s^* \propto G1$: the parameter that solves the equation *is* the gap in disguise.

Candidate C — Bilateral MGF, ratio test.

$$M(s) = 4 \cosh 3s + 2 \cosh 2s + 4 \cosh s + 2, \quad M''(s) = 36 \cosh 3s + 8 \cosh 2s + 4 \cosh s. \quad (10)$$

$M''(s)/M(s)$ is even and increases monotonically from 4 (at $s = 0$) to 9 (as $|s| \rightarrow \infty$, dominated by $e^{3|s|}$). The target $\Omega/432 \approx 0.997 < 4$, so no real s^* satisfies $M''(s^*)/M(s^*) = \Omega/432$. At $s = 0$, the ratio $M''(0)/M(0) = 4.000$, which is within 1% of $\Omega/9/N \approx 3.986$ only because $G1 < 1\%$. **PASS at $s = 0$, trivially; no proper s^* exists.**

Candidate D — Heat-kernel smearing on $\mathbb{R}$. The heat-kernel second moment shifts as $M_2(t) = 48 + 2t$. Solving $48 + 2t^* = \Omega/9$:

$$t^* = \frac{1}{2} \left(\frac{\Omega}{9} - 48 \right) = -\frac{432 - \Omega}{18} \approx -0.08265. \quad (11)$$

Since $\Omega < 432$, $t^* < 0$. Diffusion always *adds* variance; a negative diffusion time is unphysical. **PASS by construction, but unphysical.**

Candidate E — Uniform measure on a circle of radius R . Replacing the 12 point masses with a continuous uniform measure on a circle of radius R gives a per-root second moment $R^2/2$. The condition $N \cdot R^2/2 = \Omega$ yields $R^2 = \Omega/6 = \pi\alpha^{-1}/6$, a transcendental number with no simple form. The variant requiring $N \cdot R^2/2 = \Omega/9$ gives $R^2 = \Omega/54$, equally transcendental and by construction. **PASS by construction; no predictive content.**

Candidate F — Weyl character at imaginary argument. The G_2 adjoint Weyl character ($\dim = 14$) at group element e^{isH_s} is

$$\chi_{\text{adj}}(s) = 4 \cos 3s + 2 \cos 2s + 4 \cos s + 4. \quad (12)$$

$\chi_{\text{adj}}(0) = 14$, $\chi_{\text{adj}}(\pi) = -2$; the unique zero in $[0, \pi]$ is at $s_0 \approx 0.8812\pi$. Values at $s = \pi/n$ for $n = 1, \dots, 12$ are listed in Table ???. The character crosses $\alpha = \alpha^{-1,-1} \approx 7.297 \times 10^{-3}$ at $s^* \approx 0.8809\pi$, near but not equal to the zero. No ratio s^*/π matches a simple fraction to four significant figures. **PASS in the sense that a crossing exists, but the target (α) is not the second-moment target $\Omega/9$; the candidates are incommensurable.**

n	$s = \pi/n$	$\chi_{\text{adj}}(\pi/n)$	n	$s = \pi/n$	$\chi_{\text{adj}}(\pi/n)$
1	π	-2	7	$\pi/7$	+9.741
2	$\pi/2$	+2	8	$\pi/8$	+10.640
3	$\pi/3$	+1	9	$\pi/9$	+11.291
4	$\pi/4$	+4	10	$\pi/10$	+11.773
5	$\pi/5$	+6.618	11	$\pi/11$	+12.140
6	$\pi/6$	+8.464	12	$\pi/12$	+12.424

Table 1: Weyl character $\chi_{\text{adj}}(\pi/n)$ for G_2 . Note $\chi_{\text{adj}}(\pi/5) = 4 + \sqrt{5} \cdot \frac{6+\sqrt{5}}{4} \approx 4 + 2.618$; the golden ratio enters but does not produce α^{-1} .

3 Script output (verbatim)

```
=====
P154 SPECTRAL LAPLACE VERIFICATION
=====
ALPHA_INV      = 137.0363037759
BREATH_PERIOD  = 430.5122452174
BREATH_PERIOD/9 = 47.8346939130    <- transcendental target
J_discrete     = 48                <- Killing form (integer)
G1 = (432-BP)/BP = 3.45577809e-03  (0.345578%)

CANDIDATE A | Gaussian smearing sigma^2 = pi
M2_A = 48 + 12*pi = 85.69911184
Candidate A: result=85.69911184, target=47.83469391, ratio=1.79156810, FAIL

CANDIDATE B | Laplace MGF L(s) = Sum_alpha exp(-s*<alpha,H_s>^2)
L(0)=12.0000, lim_inf L(s)=2.0000
-L'(0)/L(0) = 4.000000 = J_discrete/N = 48/12 = 4
Seek s* where -L'(s*)/L(s*) = BP/108 = 3.9862244928
s* = 0.0009847302
s* / (1/BREATH_PERIOD) = 0.423938
s* / (pi/432)           = 0.135410
d(-L'/L)/ds|_{s=0}     = -14.000000
```

Linear estimate $s^* \sim (4 - BP/108)/14 = (432-BP)/1512$
 $s^* * BREATH_PERIOD = 0.42393842$
 $s^* * 432 = 0.42540346$
Candidate B: result=47.83469391, target=47.83469391, ratio=1.00000000, PASS
[$s^* = 9.847e-04$, proportional to G_1 ; no independent geometric form]

CANDIDATE C | Bilateral MGF $M(s)=4\cosh(3s)+2\cosh(2s)+4\cosh(s)+2$
 $M(0)=12$, $M''(0)=48$, $M''(s)/M(s)$ in $[4, 9]$ (monotone increasing)
Target $BP/432=0.99656 < 4$: no real s^* satisfies $M''/M = BP/432$
Alt target $TARGET/N = 3.98622 < 4$: no s^* for M''/M either
Candidate C: result=4.00000000, target=3.98622449, ratio=1.00345578, PASS
[at $s=0$ only; passes because $G_1 < 1\%$]

CANDIDATE D | Heat-kernel smearing $M_2_D(t) = 48 + 2t$
 $t^* = (BP/9 - 48)/2 = -0.0826530435$
 $t^* = -(432-BP)/18 = -0.0826530435$
 $t^* < 0$: unphysical (diffusion adds variance)
Candidate D: result=47.83469391, target=47.83469391, ratio=1.00000000, PASS
[by construction; unphysical $t^* < 0$]

CANDIDATE E | Uniform circular measure; $N*R^2/2 = BREATH_PERIOD$
 $R^2 = BP/6 = 71.7520408696$
 $R = \sqrt{\pi*\alpha_inv/6} = 8.4706576409$
 $R^2/\pi = ALPHA_INV/6 = 22.83938396$ (transcendental)
Candidate E: result=430.51224522, target=430.51224522, ratio=1.00000000, PASS
[by construction; target is BP, not BP/9]
For BP/9 version: $R^2 = BP/54 = 7.9724\dots$, also by construction.

CANDIDATE F | Weyl character $\chi(s)=4\cos(3s)+2\cos(2s)+4\cos(s)+4$
 $\chi(0)=14$ (=dim G_2), $\chi(\pi)=-2$
Zero in $[0,\pi]$: $s_zero = 2.7683344491$ ($s/\pi = 0.88119$)
Crossing $\chi(s^*) = \alpha = 7.297e-3$ at $s^* = 2.7675701265$ ($s^*/\pi = 0.88094$)
Candidate F: result=0.00729734, target=0.00729734, ratio=1.00000000, PASS
[crossing exists; but target is α , not BP/9]

SUMMARY

$G_1 = (432-BP)/BP = 3.45577809e-03$
 $BREATH_PERIOD/9 = 47.8346939130$
 $J_discrete = 48$ (integer)
 $48 - BP/9 = 0.1653060870$ (size of gap in moment space)

Candidate A: FAIL [48+12pi = 85.7 vs target 47.83]
Candidate B: PASS [s^* proportional to G_1 ; no new geometric content]
Candidate C: PASS [at $s=0$; no s^* for exact match; $G_1 < 1\%$ gives free ride]
Candidate D: PASS [unphysical $t^* < 0$; trivial by construction]
Candidate E: PASS [by construction; $R^2 = BP/54$]
Candidate F: PASS [crossing of α ; different target from BP/9]

CONCLUSION: No candidate naturally closes G_1 .

$G_1 = (432 - \pi \cdot \alpha_{\text{inv}}) / (\pi \cdot \alpha_{\text{inv}})$ requires transcendental input beyond G_2 geometry. The gap 0.345578% remains open after P154.

4 Findings and implication for G1

What passes and why

Of the six candidates, only Candidate A genuinely *tests* a specific physical hypothesis ($\sigma^2 = \pi$) and fails decisively. The remaining passes are of three qualitatively distinct types:

Tautological (D, E) The smearing parameter is fixed by the equation result = target itself, so the pass is definitional.

G1-free-ride (C) The bilateral MGF ratio $M''(0)/M(0) = 4$ deviates from the target $\Omega/108 \approx 3.9862$ by exactly $1 + G_1$, so any check with tolerance $> 0.35\%$ trivially passes. No smearing brings the ratio below 4.

Proportional to G1 (B) The MGF root s^* exists numerically and equals $(432 - \Omega)/1512$ to leading order. The numerator is G_1 times Ω . Far from providing geometric input, s^* *encodes* the gap — it is not an independent derivation of G_1 .

Different target (F) The Weyl character crossing at $s^* \approx 0.881\pi$ locates the fine-structure constant α in the character spectrum, which is a genuinely interesting fact about G_2 , but its target (α , not $\Omega/9$) is incommensurable with the second-moment question.

Why no Laplace smearing closes G1

The root of the difficulty is algebraic. The discrete second moment is the integer 48. The target $\Omega/9 = \pi(4\pi^3 + \pi^2 + \pi)/9$ is a specific transcendental combination of π . The difference

$$48 - \frac{\Omega}{9} = 48 - \frac{\pi(4\pi^3 + \pi^2 + \pi)}{9} = 48 - \frac{4\pi^4 + \pi^3 + \pi^2}{9} \approx 0.1653 \quad (13)$$

is itself transcendental. A Laplace smearing of μ_{disc} modifies the second moment by an amount that depends on the smearing parameter s and the root positions, both of which are algebraic (the root projections are integers). No algebraic function of integer data can produce the transcendental correction $(432 - \Omega)/18$ without *additional transcendental input*.

The additional input required is precisely the identification $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$: once that identity is granted, $\Omega/9$ is determined, and the “gap” is zero by definition. What is missing is not a smearing prescription but a *derivation of why* the coroot second moment 48 and the continuous breath period Ω are related by the factor 9 up to the residual $\approx 0.345\%$. Such a derivation would need to originate from a continuous geometric structure that connects the G_2 root system to S^3 holonomy at the TOE scale — a layer of geometry not present in the discrete Killing form alone.

What would be needed

Closing G_1 via spectral methods would require one of the following:

1. A continuous measure on the G_2 root system (e.g. the Plancherel measure of the associated Lie group) whose second moment evaluates to $\Omega/9$ *from first principles*, not by setting a free parameter.
2. A spectral identity of the form $\int_{G_2} \langle \alpha, H_s \rangle^2 d\mu_{\text{Haar}} = c \cdot \Omega$ for an integer c derived from representation theory.

3. An embedding argument showing that the G_2 coroot projections, when lifted to S^3 via the TOE Hopf lapse, acquire a continuous spectral density whose second moment is $\Omega/9$ by the same identities that produce $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$.

None of these is established. The Weyl character result (Candidate F) hints that G_2 's character spectrum makes contact with α at a specific imaginary argument $\approx 0.881\pi$, which is suggestive of path (3), but the connection to the second-moment question is not yet explicit.

Status of G1

G1 remains open. The discrete G_2 root Killing form ($= 48$) and the continuous breath-period second moment ($\Omega/9 \approx 47.835$) are related by $G1 \approx 0.3456\%$, a transcendental gap that no Laplace-type smearing of the discrete measure closes without importing the very transcendental it is supposed to derive.

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