

$B_{F_4}(H_s, H_s)$: Is It $\pi \cdot \alpha^{-1}$?

Addendum 153 to the TOE Corpus

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Abstract

Addendum 151 established $B_{G_2}(H_s, H_s) = 48$ and showed that the product $J_{\text{short}} \times J_{\text{long}} = 432 \approx \pi \alpha^{-1}$. The natural next question is whether the ambient F_4 Killing form, evaluated on the same coroot H_s , equals $\text{BREATH_PERIOD} = \pi \cdot \alpha^{-1} \approx 430.51$ directly. We compute $B_{F_4}(H_s, H_s)$ in the standard F_4 root system in $\mathbb{R}^4$ (48 roots: 24 short of length 1, 24 long of length $\sqrt{2}$) with $H_s = e_1 - e_2$ (the coroot of the long root $e_1 - e_2$, natural embedding of the G_2 short coroot). The result is $B_{F_4}(H_s, H_s) = 36$ (exact integer). G1 does not close via this route.

1 Setup and Computation

1.1 F_4 root system

In the standard $\mathbb{R}^4$ realisation (Bourbaki):

$$\Phi_{\text{short}} = \{\pm e_i\}_{i=1}^4 \cup \frac{1}{2}\{(\pm 1, \pm 1, \pm 1, \pm 1)\}, \quad |\alpha|^2 = 1, \quad |\Phi_{\text{short}}| = 24, \quad (1)$$

$$\Phi_{\text{long}} = \{\pm e_i \pm e_j\}_{i < j}, \quad |\alpha|^2 = 2, \quad |\Phi_{\text{long}}| = 24. \quad (2)$$

Total $|\Phi_{F_4}| = 48$.

1.2 Killing form and coroot

The Killing form (adjoint trace) is

$$B_{F_4}(H, K) = \sum_{\alpha \in \Phi_{F_4}} \alpha(H) \alpha(K), \quad (3)$$

where $\alpha(H) = 2\langle \alpha, H \rangle / |\alpha|^2$ for the coroot pairing.

We take $H_s = e_1 - e_2 \in \mathbb{R}^4$, the coroot of the long root $e_1 - e_2$ ($|\alpha|^2 = 2$, so $\alpha^\vee = \alpha$). Under the standard $G_2 \subset B_3 \subset F_4$ embedding, this vector represents the G_2 short coroot in F_4 coordinates (see Addendum 151 for the G_2 context).

Then $\alpha(H_s) = \langle \alpha, e_1 - e_2 \rangle$ for all long roots, and $\alpha(H_s) = 2\langle \alpha, e_1 - e_2 \rangle$ for all short roots. Squaring and summing:

$$B_{F_4}(H_s, H_s) = \sum_{\alpha \in \Phi_{\text{long}}} \langle \alpha, e_1 - e_2 \rangle^2 + \sum_{\alpha \in \Phi_{\text{short}}} 4\langle \alpha, e_1 - e_2 \rangle^2. \quad (4)$$

2 Result

2.1 Python output

```
Short roots: 24, Long roots: 24
H_s = [ 1 -1  0  0]
B_F4(H_s,H_s) = 36.0
B_G2_check = 16 (expect 48 from P151)
BREATH_PERIOD = pi*alpha^{-1} = 430.51224522
Gap G1: (432 - BREATH_PERIOD)/BREATH_PERIOD = 0.3456%
B_F4 vs BREATH_PERIOD: diff = -394.512245, relative = -91.6379%
G1 closed by F4? False
```

--- Additional analysis ---

```
B_F4 = 36.0
B_F4 as integer? 36 (diff from int: 0.00e+00)
B_F4 / B_G2 (P151 = 48) = 0.750000
B_F4 / 432 = 0.083333
B_F4 / BREATH_PERIOD = 0.083621
```

Decomposition:

```
B_F4 from short F4 roots: 12.0
B_F4 from long F4 roots: 24
B_F4_long / B_F4_short = 2.0000
```

2.2 Numerical verdict

$$\boxed{B_{F_4}(H_s, H_s) = 36} \quad \text{vs} \quad \pi \cdot \alpha^{-1} = 430.512\dots \quad (5)$$

The relative gap is 91.6%. G1 does *not* close.

3 Structural observations

3.1 Sector decomposition

Sector	$ \Phi $	$\sum \alpha(H_s)^2$	ratio
F_4 short ($ \alpha ^2 = 1$)	24	12	—
F_4 long ($ \alpha ^2 = 2$)	24	24	$2\times$
Total $B_{F_4}(H_s, H_s)$	48	36	—
G_2 (P151, Cartan matrix)	12	48	—

Three coincidences are worth noting without overweighting them:

- $B_{F_4}(H_s, H_s) \cdot J_{\text{short}}(G_2) = 36 \times 12 = 432 \approx \text{BREATH_PERIOD}$. The F_4 total and the G_2 short Jacobian (P151) multiply to the integer approximant of BREATH_PERIOD.
- $B_{F_4}(H_s, H_s) \cdot B_{G_2}(H_s, H_s) = 36 \times 48 = 1728 = 12^3$. The product of the two ambient-algebra Killing form values is a perfect cube.
- The F_4 short sector contributes $12 = J_{\text{short}}(G_2)$ exactly, and the long sector contributes $24 = 2J_{\text{short}}$. The long/short ratio is 2, not $3 = C_A$ (cf. P151 §4).

3.2 Embedding note

The script's G_2 -consistency check recovers $B_{G_2} = 16$ rather than 48. The factor-of-3 discrepancy confirms that the Euclidean inner product on $\Phi_{F_4} \cap \mathbb{R}^3$ (the long roots of F_4 lying in the first three coordinates) is *not* the same as the G_2 Killing form pairing: the G_2 long roots embedded in this way have $|\alpha|_{\mathbb{R}^4}^2 = 2$ while the abstract G_2 normalisation uses $|\alpha_{\text{long}}|^2 = 3$, introducing an unresolved rescaling factor of $3/2$. The abstract $B_{G_2}(H_s, H_s) = 48$ of P151 uses the G_2 Cartan matrix and is the correct Lie-algebraic value; the F_4 ambient computation and the G_2 subalgebra computation live in different normalisations and do not add naively.

4 Open items

- P153-O1. Correct embedding.** Identify the precise image of the G_2 short coroot H_s in F_4 coordinates under the standard $G_2 \subset F_4$ embedding (fixed-point of the order-3 automorphism of D_4 , or $G_2 \subset B_3 \subset F_4$) and recompute $B_{F_4}(H_s, H_s)$. It is not excluded that the correct embedded H_s yields a value related to BREATH_PERIOD by a Dynkin-index factor.
- P153-O2. Product structure.** The identity $B_{F_4}(H_s, H_s) \times J_{\text{short}}(G_2) = 432 \approx \text{BREATH_PERIOD}$ and the cube $B_{F_4} \times B_{G_2} = 12^3$ are structural coincidences at the integer-approximant level. Determine whether they survive under the corrected embedding of O1 and whether $12^3/4 = 432$ has a TOE derivation.
- P153-O3. Dynkin index route.** Compute the Dynkin index ℓ of $G_2 \hookrightarrow F_4$ and verify $B_{F_4}|_{G_2} = \ell \cdot B_{G_2}$. Check whether $\ell \cdot 48$ lands near BREATH_PERIOD for any natural choice of ℓ .

5 Conclusion

$B_{F_4}(H_s, H_s) = 36$. The claim $B_{F_4}(H_s, H_s) = \pi\alpha^{-1}$ is false for the natural embedding. G1 remains open. The integer 36 relates to BREATH_PERIOD only via the product $36 \times 12 = 432$ (the integer approximant of BREATH_PERIOD found in P151–152); no direct algebraic route from the F_4 Killing form to $\pi\alpha^{-1}$ is apparent from this computation. Open items P153-O1 through O3 define the next search.

Script: Lumen/corpus/addenda/verify/verify_P153.py