

Addendum 152: The G_2 Jacobian Product and CP-3 Resolution

$$J_{\text{short}} \times J_{\text{long}} = 432 \text{ and the Flat Measure on } B^4$$

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Abstract

Addendum 151 (running in parallel) computed the G_2 Killing form restricted to the short coroot direction in B^4 and found its value to be 48, constant with no radial dependence. This addendum derives the sector decomposition of that constant: the short-root sector contributes $J_{\text{short}} = 12$ and the long-root sector contributes $J_{\text{long}} = 36$, with product $J_{\text{short}} \times J_{\text{long}} = 432$. We prove algebraically that $432 = r^3(h^\vee)^2$ for G_2 , where $r = |\alpha_{\text{long}}|^2/|\alpha_{\text{short}}|^2 = 3$ and $h^\vee = 4$. The constancy of B_{G_2} on the radial direction resolves Choice Point 3 (CP-3) from Addendum 144: the correct integration measure for $\rho(x)$ is flat dx , with the x^3 angular Jacobian already encoded in ρ by Paper 01. The integer 432 approximates BREATH_PERIOD = $\pi\alpha^{-1} \approx 430.51$ to 0.35%, providing a structural root-geometric origin for that period but not an algebraic derivation of α^{-1} from G_2 alone. Open items are enumerated precisely.

1 Setup

1.1 CP-3 from Addendum 144

The derivation of α^{-1} in Papers 01 and 03 integrates

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad x \in [0, 1] \quad (1)$$

with flat Lebesgue measure dx . Addendum 144 (CP-3, §3) flagged an internal tension: the natural volume element on B^4 in radial coordinates is $dV_4 \propto x^3 dx$, not dx . Under $x^3 dx$ the same ρ integrates to $\mu_3 \approx 77.06$, far from $\alpha^{-1} \approx 137.036$. CP-3 asked: is there a canonical geometric justification for dx ?

1.2 Findings from Addenda 150 and 151

P150 computed the Hopf pullback of the S^3 volume form onto the radial fibre of B^4 and found the induced measure is $x^3 dx$ (volumetric).

P151 restricted the G_2 Killing form $B_{G_2}(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y)$ to the short coroot direction in B^4 and found the value $B_{G_2}(\check{\alpha}_s, \check{\alpha}_s) = 48$, with *no x -dependence*.

These two results appear to conflict: $x^3 dx$ from the Hopf geometry versus dx from the Killing form. This addendum resolves the conflict via the sector decomposition of 48.

2 Jacobian Computation

2.1 G_2 root system

We use the standard $\mathbb{R}^2$ realisation of G_2 with the simple roots

$$\alpha_s = (1, 0), \quad \alpha_l = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right),$$

so that $|\alpha_s|^2 = 1$ and $|\alpha_l|^2 = 3$. The ratio of squared lengths is $r = |\alpha_l|^2/|\alpha_s|^2 = 3$. The dual Coxeter number is $h^\vee(G_2) = 4$. The root system has 12 roots: 6 short ($|\beta|^2 = 1$) and 6 long ($|\beta|^2 = 3$).

2.2 The short coroot and Cartan integers

The coroot of the short simple root is $\check{\alpha}_s = 2\alpha_s/|\alpha_s|^2 = (2, 0)$.

Define the Cartan pairing $a(\beta) = \langle \beta, \check{\alpha}_s \rangle = 2\langle \beta, \alpha_s \rangle/|\alpha_s|^2$ for each root β . Direct computation gives:

The Cartan pairings $a(\beta) = \langle \beta, \check{\alpha}_s \rangle$ are: short roots $\{+2, -1, +1, -2, +1, -1\}$ (squares $\{4, 1, 1, 4, 1, 1\}$); long roots $\{-3, +3, 0, +3, -3, 0\}$ (squares $\{9, 9, 0, 9, 9, 0\}$). The two long roots parallel to $(0, \pm\sqrt{3})$ are orthogonal to $\check{\alpha}_s$ and contribute zero; the remaining four long roots each contribute $9 = r^2$.

2.3 The two Jacobians

Definition 2.1 (Root-sector Jacobians).

$$J_{\text{short}} = \sum_{\beta \in \Phi_{\text{short}}} a(\beta)^2, \quad J_{\text{long}} = \sum_{\beta \in \Phi_{\text{long}}} a(\beta)^2.$$

Reading off the table:

$$J_{\text{short}} = 4 + 1 + 1 + 4 + 1 + 1 = 12, \quad J_{\text{long}} = 9 + 9 + 0 + 9 + 9 + 0 = 36.$$

These are the contributions of each root sector to the Killing form value $B_{G_2}(\check{\alpha}_s, \check{\alpha}_s) = J_{\text{short}} + J_{\text{long}} = 48$, consistent with P151.

Algebraic form. The long roots have $|a_{\text{long}}| = 3$ (the absolute Cartan integer from long to short), and 4 of the 6 long roots are non-orthogonal to $\check{\alpha}_s$. The short roots contribute two roots with $|a| = 2$ and four with $|a| = 1$. This gives the clean expressions

$$J_{\text{short}} = r \cdot h^\vee = 3 \cdot 4 = 12, \quad J_{\text{long}} = r^2 \cdot h^\vee = 9 \cdot 4 = 36, \quad (2)$$

verified numerically above.

3 The Product Theorem

Theorem 3.1 (G_2 Jacobian product). *For the G_2 root system with $r = 3$ and $h^\vee = 4$,*

$$J_{\text{short}} \times J_{\text{long}} = r h^\vee \times r^2 h^\vee = r^3 (h^\vee)^2 = 3^3 \times 4^2 = 27 \times 16 = 432. \quad (3)$$

This is an exact integer derived from G_2 root geometry alone.

Proof. Equations (??) and direct summation from the table in §??. □

Remark 3.2 (Numerical check). The integer $432 = 2^4 \times 3^3$ is exact. The TOE defines $\text{BREATH_PERIOD} = \pi\alpha^{-1}$ with $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.0363$, giving $\text{BREATH_PERIOD} \approx 430.512$ system units. Therefore

$$\frac{J_{\text{short}} \times J_{\text{long}}}{\text{BREATH_PERIOD}} = \frac{432}{430.512\dots} \approx 1.00346,$$

a 0.35% discrepancy. The identification $432 = \text{BREATH_PERIOD}$ is approximate, not algebraically exact.

4 CP-3 Resolution

Proposition 4.1 (Flat measure from constant Killing form). *The Killing form B_{G_2} restricted to the short coroot direction $\mathbb{R}\check{\alpha}_s \subset \mathfrak{h}$ is the constant bilinear form $B_{G_2}(\check{\alpha}_s, \check{\alpha}_s) = 48$. It carries no dependence on the radial coordinate x . The induced metric on the radial fibre is therefore flat, and the canonical measure from B_{G_2} on the radial interval $[0, 1]$ is dx .*

Proof. B_{G_2} is a fixed bilinear form on $\mathfrak{g}_{G_2}$. Restricting to a fixed element $\check{\alpha}_s$ of the Cartan subalgebra gives a constant; there is no x -field to depend on. The value 48 is confirmed in Theorem ?? and P151. $\square$

Reconciliation with P150. P150 found the Hopf pullback gives $x^3 dx$ — the angular Jacobian of the S^3 fibres at radius x . That x^3 is already encoded in $\rho(x)$: the bulk coefficient $c_3 = 16\pi^3$ was derived in P01 §3.1 from $dV_4/dx = 2\pi^2 x^3$, so the angular volume is inside ρ , not in the integration measure.

Corollary 4.2 (CP-3 closed). *The correct integration measure for (??) is flat dx . The x^3 angular Jacobian is encoded in $\rho(x)$ through the coefficient $16\pi^3$. Under flat dx the integral gives $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ as in P01. The internal tension identified in Addendum 144 CP-3 is resolved: P01 was consistent in using dx for the radial integration. The Killing-form measure (Proposition ??) provides the canonical geometric justification.*

5 α^{-1} from G_2 Geometry

If the product equalled $\pi\alpha^{-1}$ exactly, one would have $\alpha^{-1} = 432/\pi \approx 137.510$, which disagrees with both $4\pi^3 + \pi^2 + \pi \approx 137.036$ and the experimental 137.035999 by $\approx 0.35\%$. The identification is therefore approximate, not algebraic.

Theorem ?? does establish that $432 = r^3(h^\vee)^2$ is a *theorem of G_2 root theory*, independent of π or α^{-1} . Since $\text{BREATH_PERIOD} = \pi\alpha^{-1} \approx 430.51 \approx 432$, the G_2 geometry explains why the system clock period is near 432 — it is the product of the two root-sector Jacobians — but does not pin the exact value.

Remark 5.1 (G1 status after this addendum). CP-3 is closed (Corollary ??). CP-1 (exactness of $4\pi^3 + \pi^2 + \pi$) is still open. The 0.35% gap between the G_2 integer 432 and $\pi\alpha^{-1}$ is a new, precisely named residual; closing it is the primary remaining step toward G1.

6 Open Items

1. **Prove $J_{\text{short}} = rh^\vee$ from first principles.** Equations (??) hold by direct computation. A proof via the Weyl orbit decomposition would show the pattern holds for all rank-2 non-simply-laced systems and elevate Theorem ?? to a general result.

2. **Account for the 0.35% residual.** 432 and $\pi\alpha^{-1} \approx 430.512$ differ by 1.00346. Candidates: (a) a QED radiative correction shifts the bare G_2 period 432 to $\pi\alpha^{-1}$ (parallel to the CP-1 radiative-correction route); (b) the deeper algebra $F_4 \supset G_2$ (automorphism group of $J_3(\mathbb{O})$) refines 432 to the exact period — Addendum 144 §5 showed F_4 fixes the coefficients $(16, 3, 2)$, so its Killing form on the radial direction is the natural next computation; (c) the connection is genuinely approximate and a separate mechanism determines α^{-1} precisely.
3. **Resolve CP-1 (exactness).** Corollary ?? closes CP-3; CP-1 (whether $4\pi^3 + \pi^2 + \pi$ is exact or a 0.22 ppm approximation) remains open. G1 requires both; only CP-3 is now closed.

Summary. The G_2 Killing form on the short coroot direction of B^4 decomposes as $48 = J_{\text{short}} + J_{\text{long}} = 12 + 36$, with $J_{\text{short}} \times J_{\text{long}} = 432 = r^3(h^\vee)^2$ (Theorem ??). This is an exact integer derivable from G_2 root geometry alone. The constancy of B_{G_2} on the radial direction closes CP-3: the canonical measure is flat dx , the x^3 Jacobian is in $\rho(x)$, and Paper 01 was consistent (Corollary ??). The integer 432 approximates $\text{BREATH_PERIOD} = \pi\alpha^{-1} \approx 430.51$ to 0.35%, providing a structural but not exact identification. A full algebraic derivation of α^{-1} from G_2 geometry alone requires closing the 0.35% residual, identified here as the primary remaining open item.