

G₂ Angle Uniqueness and the Hopf Measure Pullback

Addendum 150 to the TOE Corpus

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2026-05-15

Abstract

Addendum 132 derived the matching scale $\mu_{\text{match}} = m_{\text{conf}} \cdot \exp(2\pi(6 - \sqrt{3})/23)$ from the G₂ root fan and identified $\theta = \pi/6$ as the angle between the short-root (SU(3) colour) direction and the long-root (G₂ extra) direction, leaving uniqueness as an open item. This addendum proves that $\theta = \pi/6$ is uniquely forced by two constraints that admit no freedom: (i) the G₂ root system has exactly 12 roots, equally spaced at $\pi/6$ in the root plane; and (ii) the embedding $\text{SU}(3) \subset \text{G}_2$ identifies the six short roots with the A_2 root system, fixing their angular positions. Together these leave the long-root fan at precisely $\pi/6$ offset from the short-root fan — no choice is involved. A second result computes the pullback of the G₂-invariant (round) measure on S^3 to the radial interval $[0, 1]$ of B^4 via the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$: the pullback weight is $x^3 dx$ (volumetric), not flat dx . The implication for CP-3 (P144): the G₂ boundary geometry supports the coefficient 16 via the volumetric measure $dV_4 = 2\pi^2 x^3 dx$, but does not resolve the P01 inconsistency. The F₄-flat-measure hypothesis of CP-3 requires an independent $J_3(\mathbb{O})$ argument.

1 Setup: open items from Addendum 132 and CP-3

Addendum 132 showed that every factor in

$$\mu_{\text{match}} = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right) \quad (1)$$

is TOE-native: $\sqrt{3} = \rho_{\text{G}_2}$ is the G₂ long-to-short root ratio, $6 = |\Phi^+(\text{G}_2)| = 2\rho_{\text{G}_2}^2$ counts the positive roots of G₂, and $23 = 3b_0$ comes from the $J_3(\mathbb{O})$ Casimir. The derivation used the single angle $\theta = \pi/6$ for both the non-perturbative boundary value $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$ and the matching value $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$. Addendum 132 noted that θ is the half-opening of the root fan under the SU(3) subgroup, but did not prove this is uniquely forced.

The open problem CP-3 from Addendum 144 concerns a different but related inconsistency in Paper 01. Paper 01 motivates the coefficient 16 in the cubic phase density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ from the B^4 volume element $dV_4 = 2\pi^2 r^3 dr$, since $\int_0^1 16\pi^3 x^3 dx = 4\pi^3$ requires the flat integration dx while the motivating geometry uses $2\pi^2 x^3 dx$. Addendum 144 proposed that the F₄-invariant inner product on $J_3(\mathbb{O})$ might force the radial measure to be flat dx , resolving the mismatch. The G₂ angle θ couples the S^3 boundary geometry to the B^4 interior; if θ is uniquely determined, it constrains whether the boundary-to-interior coupling transforms the measure.

This addendum answers both questions.

2 The G_2 root system

The exceptional Lie algebra G_2 has rank 2. Its root system $\Phi(G_2)$ consists of 12 roots in a two-dimensional real plane, comprising 6 short roots of length $s = 1$ and 6 long roots of length $l = \sqrt{3}$, giving root-length ratio $\rho_{G_2} = l/s = \sqrt{3}$.

Choose simple roots α_1 (short) and α_2 (long). The Cartan matrix entries are $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -3$, encoding the angle between the simple roots:

$$\cos \angle(\alpha_1, \alpha_2) = \frac{\langle \alpha_1, \alpha_2 \rangle}{|\alpha_1||\alpha_2|} = \frac{-3/2}{1 \cdot \sqrt{3}} = -\frac{\sqrt{3}}{2}, \quad \angle(\alpha_1, \alpha_2) = \frac{5\pi}{6}. \quad (2)$$

The six positive roots, together with their lengths, are:

$$\begin{aligned} &\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \quad (\text{short, length } 1), \\ &\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \quad (\text{long, length } \sqrt{3}). \end{aligned}$$

Appending their negatives gives all 12 roots. A direct calculation (using $\langle \alpha_1, \alpha_2 \rangle = -3/2$) verifies each length: $|\alpha_1 + \alpha_2|^2 = 1 - 3 + 3 = 1$, $|2\alpha_1 + \alpha_2|^2 = 4 - 6 + 3 = 1$, $|3\alpha_1 + \alpha_2|^2 = 9 - 9 + 3 = 3$, $|3\alpha_1 + 2\alpha_2|^2 = 9 - 18 + 12 = 3$.

The Weyl group $W(G_2)$ is the dihedral group of order 12, acting by reflections in the 6 root hyperplanes (lines through the origin perpendicular to each positive root). This produces 12 Weyl chambers, each a sector of angular width $2\pi/12 = \pi/6$ in the root plane.

3 Uniqueness of $\theta = \pi/6$

Theorem 3.1 (Uniqueness of θ). *Let G_2 be the rank-2 exceptional root system with the embedding $SU(3) \subset G_2$ that identifies the six short roots of G_2 with the A_2 ($SU(3)$) root system. Then the angle between any short root and its nearest long root is uniquely $\theta = \pi/6$. No continuous freedom remains; θ is not a parameter.*

Proof. Step 1: the 12 roots are equally spaced. The Weyl group $W(G_2) \cong \text{Dih}_{12}$ is the dihedral group of order 12, acting faithfully on the root plane. Its 12 elements map roots to roots and include a rotation by $2\pi/12 = \pi/6$. Since the rotation by $\pi/6$ is an element of $W(G_2)$ and the root system is $W(G_2)$ -stable, the 12 roots must be permuted transitively (up to the short/long distinction) by this rotation. Concretely, the 12 roots occupy positions at angles $\phi_k = \phi_0 + k\pi/6$ for $k = 0, 1, \dots, 11$ in the root plane, where ϕ_0 is the angle of any fixed root.

Step 2: $SU(3)$ fixes the short-root positions. The six short roots of G_2 form an A_2 root system with $|W(A_2)| = 6$; they are equally spaced at $\pi/3$. The embedding $SU(3) \subset G_2$ identifies these six roots with the colour-sector root system and fixes their angular positions:

$$\phi_k^{\text{short}} = \phi_0 + k \cdot \frac{\pi}{3}, \quad k = 0, \dots, 5. \quad (3)$$

The orientation ϕ_0 is a global frame choice (rotation of the entire root diagram), which is immaterial for the angle θ between adjacent roots of different type.

Step 3: the long roots fill the remaining positions. From Step 1, the 12 positions are $\{\phi_0 + k\pi/6 : k = 0, \dots, 11\}$. The six short roots occupy the even positions $k = 0, 2, 4, 6, 8, 10$ (i.e. spacing $\pi/3$ as in Step 2). The six long roots therefore occupy the odd positions $k = 1, 3, 5, 7, 9, 11$, giving angular positions

$$\phi_k^{\text{long}} = \phi_0 + \frac{\pi}{6} + k \cdot \frac{\pi}{3}, \quad k = 0, \dots, 5. \quad (4)$$

The offset between adjacent short and long positions is $\phi_k^{\text{long}} - \phi_k^{\text{short}} = \pi/6$.

Step 4: no freedom. The angle $\theta = \pi/6$ between consecutive short and long roots is completely fixed by (a) the 12-fold Dih_{12} symmetry of $W(\text{G}_2)$ and (b) the six-fold placement of the short roots forced by $\text{SU}(3) \subset \text{G}_2$. There is no continuous parameter to choose; any deformation of θ would violate the $W(\text{G}_2)$ root axioms or the $\text{SU}(3)$ embedding. $\square$

Remark 3.2. The uniqueness is stronger than just fixing an angle: it fixes the entire interleaved short/long root fan. The angle $\theta = \pi/6$ is the angular width of a G_2 Weyl chamber, and each such chamber is bounded on one side by a short-root hyperplane and on the other by a long-root hyperplane. The Weyl chamber width $\pi/6$ is a discrete invariant of the root system, not a metric parameter.

Corollary 3.3. *The exponent in equation (??) inherits the full rigidity of the G_2 root system. The two trigonometric values at $\theta = \pi/6$ used in Addendum 132 — $\cot(\pi/6) = \sqrt{3}$ and $\sin(\pi/6) = 1/2$ — are forced by the same geometric constraint, with no remaining freedom.*

4 Measure pullback via Hopf

Theorem 4.1 (Hopf pullback of the G_2 -induced S^3 measure). *Let $S^3 = \partial B^4$ carry the round ($\text{SO}(4)$ -invariant) Riemannian measure $d\sigma$, which is the unique G_2 -compatible measure on S^3 . Let $\pi_H : S^3 \rightarrow S^2$ be the Hopf fibration with S^1 fibers. Identify the radial interval $[0, 1]$ of B^4 with the orbit space of the Hopf-fibered boundary via the radial coordinate $x \in [0, 1]$. Then the measure on $[0, 1]$ induced from the B^4 volume form $dV_4 = 2\pi^2 x^3 dx$ is proportional to $x^3 dx$.*

Proof. The B^4 volume form in radial coordinates is $dV_4 = \text{vol}(S_x^3) dx = 2\pi^2 x^3 dx$, where S_x^3 is the sphere of radius x with round area $2\pi^2 x^3$. The G_2 root geometry is discrete (a finite set of reflection hyperplanes in the 2D root plane) and does not deform the Riemannian metric on $S^3 \subset \mathbb{R}^4$. The G_2 -compatible measure on S^3 is therefore the round measure $d\sigma = 2\pi^2 d(\text{normalised area})$, which is the unique $\text{SO}(4)$ -invariant probability measure on S^3 (the $\text{SU}(3) \subset \text{G}_2$ subgroup acts by isometries that preserve this measure). Under the Hopf fibration $\pi_H : S^3 \rightarrow S^2$, the round measure decomposes as

$$d\sigma_{S^3} = d\sigma_{S^2} \otimes d\theta_{S^1} \quad (\text{Riemannian submersion}), \quad (5)$$

where $d\sigma_{S^2}$ is the Fubini-Study measure on S^2 and $d\theta_{S^1}$ is the Haar measure on the S^1 fibre. Integrating over the S^1 fibre (which has length 2π in unit S^3) and then pulling back to the radial B^4 coordinate $x \in [0, 1]$ yields a measure weighted by the S^3 area $2\pi^2 x^3$, giving

$$d\mu_{\text{radial}} = \underbrace{2\pi^2 x^3}_{\text{area}(S_x^3)} dx \propto x^3 dx. \quad (6)$$

This is the volumetric measure, not flat dx . $\square$

Remark 4.2. The key step is that the G_2 root angle $\theta = \pi/6$ is a property of the discrete symmetry group acting on the root plane, not a curvature parameter of the S^3 metric. The round metric on $S^3 \subset \mathbb{R}^4$ is determined by the embedding and is G_2 -invariant; it is not distorted by the root fan geometry. Hence the pullback measure is $x^3 dx$ regardless of whether one specifies the G_2 root angle or not.

5 Implication for CP-3

CP-3 (Addendum 144) identifies the following internal inconsistency in Paper 01. The coefficient 16 in $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ satisfies $\int_0^1 16\pi^3 x^3 dx = 4\pi^3$, which is the correct B^4 bulk contribution to α^{-1} . Paper 01 motivates this coefficient from $dV_4 = 2\pi^2 r^3 dr$ (volumetric), but

the integral $\int_0^1 \rho(x) dx = 4\pi^3 + \pi^2 + \pi = \alpha^{-1}$ uses flat dx . Addendum 144 proposed that the F_4 -invariant inner product on $J_3(\mathbb{O})$ might force the radial measure to be flat dx .

Theorem ?? settles the G_2 side of this question: the G_2 boundary geometry gives the volumetric weight $x^3 dx$, in agreement with $dV_4 = 2\pi^2 x^3 dx$. This has two consequences.

(a) The coefficient 16 is consistent with the volumetric measure. The G_2 root geometry supports the chain

$$\text{short roots} \xrightarrow{\text{SU}(3) \subset G_2} S^3 \text{ round measure} \xrightarrow{\text{Hopf}} x^3 dx \xrightarrow{\int_0^1 16\pi^3 x^3 dx} 4\pi^3, \quad (7)$$

with no inconsistency at the G_2 level.

(b) The flat-measure hypothesis requires a separate argument. For $\rho(x)$ to be integrated with flat dx self-consistently, the F_4 -invariant inner product on $J_3(\mathbb{O})$ would have to *override* the G_2 boundary measure and replace $x^3 dx$ with dx . This would require an independent $F_4 \supset G_2$ geometric argument showing that the $J_3(\mathbb{O})$ inner product modifies the radial measure. The G_2 uniqueness result in Section ?? shows there is no residual freedom in θ that could absorb such a modification: the G_2 geometry is fully determined, so any flat-measure resolution must come from the F_4 level. CP-3 therefore remains open, with the constraint that its resolution cannot invoke G_2 freedom.

6 Open items

F_4 geometry and the flat-measure question. F_4 has rank 4 and contains $G_2 \times \text{SU}(3)(3)$ as a maximal subgroup. The relevant question is whether the F_4 -invariant inner product on $J_3(\mathbb{O})$ — the unique (up to scale) positive definite F_4 -equivariant form — replaces the volumetric $x^3 dx$ with flat dx when restricted to the radial B^4 direction. This requires an explicit computation of the F_4 action on the radial coordinate of $B^4 \subset J_3(\mathbb{O})$, which depends on how B^4 is embedded in the 27-dimensional Jordan algebra. No such computation is available within the current corpus.

Higher-order Weyl chambers. Theorem ?? establishes $\theta = \pi/6$ as the inter-fan angle in the 2D G_2 root plane. A finer question is whether the G_2 Weyl chamber angle has a distinguished role in the 7D ambient space (the imaginary octonions) that G_2 acts on as the automorphism group of $\mathbb{O}$. In particular, the 7D analogue of the root fan (the set of G_2 -orbit hyperplanes) might impose further constraints on the $S^3 \hookrightarrow S^6$ embedding.

Hopf lapse interpretation of θ . Addendum 132 noted that the factor 2π in equation (??) corresponds to the Hopf latitude $u = \cos(2\beta)$ double-winding. Now that $\theta = \pi/6$ is uniquely fixed, a natural question is: what is the Hopf latitude $u = \cos(2\beta)$ corresponding to the G_2 fan boundary at $\theta = \pi/6$? Setting $\beta = \theta = \pi/6$ gives $u = \cos(\pi/3) = 1/2$, which is precisely the GON half-lapse. Whether this coincidence has a structural explanation — connecting the matching condition $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2 = u$ to a specific S^3 locus — is an open geometric question.