

Addendum 144: Uniqueness of α^{-1}

A Choice-Point Analysis of the (B^4, S^3) Derivation

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Abstract

Paper 143 nominated the uniqueness of $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ as the sharpest candidate Gödel sentence (G1): no other admissible (B^4, S^3) density choice yields the same fine-structure constant. We read Papers 01 and 03 directly and locate every point at which a choice is made. Four independent choice points are identified; two cannot be justified from S^3 geometry alone. We state the minimal additional axiom needed to close the gap, assess its naturalness, and evaluate whether $J_3(\mathbb{O})$ representation theory provides the missing constraint. The outcome is a precise failure-mode report: G1 remains open, but the two load-bearing gaps are now named.

1 Setup

The G1 candidate asserts: under any admissible density on (B^4, S^3) consistent with the Hopf fibration and the three-layer ontology (bulk, boundary, edge), the integral defining α^{-1} is forced to equal $4\pi^3 + \pi^2 + \pi$.

The source derivation (P01 §2, P03 §2) proceeds as follows. Introduce the normalized radial coordinate $x \in [0, 1]$ on B^4 . Define the cubic phase density

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad (1)$$

and integrate with flat (Lebesgue) measure to obtain

$$\int_0^1 \rho(x) dx = \frac{16\pi^3}{4} + \frac{3\pi^2}{3} + \frac{2\pi}{2} = 4\pi^3 + \pi^2 + \pi = 137.036303\dots \quad (2)$$

The experimental value is $\alpha_{\text{exp}}^{-1} = 137.035999084(21)$, a discrepancy of 2.2×10^{-6} (0.00022%). P01/P03 call (??) an “exact identity” to “0.0002% precision”: the precision language is correct; the exactness language is not — this distinction is CP-1 below.

The integer coefficients $(16, 3, 2)$ are justified post hoc: $16 = 2 \cdot \dim \mathbb{O}$ from octonionic structure (four arguments, P01 §3.1); $3 = \dim(\text{adj } SU(2))$ from gauge-field representation theory (P01 §3.2); $2 = \dim(\text{fund } SU(2))$ or a $\mathbb{Z}_2$ count (P01 §3.3, two interpretations).

2 The Derivation of α^{-1}

The logical structure is a four-step construction:

1. Restrict $\rho(x)$ to a degree-3 polynomial with terms $c_n x^n$, $n \in \{1, 2, 3\}$.
2. Fix the integration measure to be flat dx on $[0, 1]$.

3. Impose $\int_0^1 \rho dx = 4\pi^3 + \pi^2 + \pi$ (exact).
4. Identify $c_n = (n+1) \cdot$ (summand for term n) algebraically.

Step 3 forces $c_3 = 16\pi^3$, $c_2 = 3\pi^2$, $c_1 = 2\pi$ by trivial algebra. Step 4 then *checks* that these integers match known algebraic structures. The derivation verifies consistency; it does not derive α^{-1} from the geometry independently.

3 Choice Point Analysis

Choice Point 3.1 (Approximate identity treated as exact). The formula $4\pi^3 + \pi^2 + \pi = 137.036303\dots$ exceeds the experimental value by $\Delta = 3.05 \times 10^{-4}$ (0.22 ppm). Two resolutions exist: (a) the formula is the *bare* theoretical value and QED radiative corrections shift it to α_{exp}^{-1} (analogous to $g-2$ corrections to a tree-level formula); or (b) the formula is an approximation to a different exact expression. The TOE does not currently decide between (a) and (b). G1 is not assessable until CP-1 is resolved.

Choice Point 3.2 (Monomial polynomial basis). The three-layer ontology motivates three graded contributions, but does not uniquely select the functional form $\{x^3, x^2, x\}$. Exponential, Bessel, or trigonometric bases all admit normalizations integrating to α^{-1} with qualitatively similar layer interpretations. The polynomial form is the minimal-degree choice consistent with the grading; minimality is a natural additional assumption but is not forced by S^3 geometry.

Choice Point 3.3 (Flat integration measure). The natural volume measure on B^4 in radial coordinates is $dV_4 \propto r^3 dr \propto x^3 dx$. Under this measure the same ρ integrates to $\mu_3 = 77.06\dots \neq \alpha^{-1}$. P01 uses the volume element in one place (Approach 4 for c_3 : $dV_4/dr = 2\pi^2 r^3$ motivates the coefficient $16\pi^3$) but then integrates ρ with flat dx . This is an internal inconsistency: the coefficient is motivated by the volume measure; the integral uses the flat measure. CP-3 is the deepest structural gap in the derivation.

Choice Point 3.4 (Edge coefficient: two distinct justifications). P01 §3.3 offers two independent motivations for $c_1 = 2\pi$: $\dim(\text{fund } SU(2)) = 2$ and $|\mathbb{Z}_2| = 2$. These are not equivalent (one is a complex representation dimension, the other a group order), and no argument selects between them. The existence of two justifications for the same number signals that neither is a true derivation.

Remark 3.5 (Coefficient 16: Approach 4 is circular). Of the four arguments for $c_3 = 16\pi^3$, Approach 4 defines R by $2\pi^2 R^3 = 16\pi^3$, i.e. $R^3 = 8\pi$, then identifies $8 = \dim \mathbb{O}$. R is chosen to make the coefficient work; the “derivation” reverses the inference. Approaches 1–3 (doubling, complexification, $SO(8)$ triality) give $16 = 2 \times 8$ without circularity and should be taken as the canonical argument.

4 Uniqueness Theorem or Failure Mode

Theorem 4.1 (Conditional uniqueness). *Assume: (A1) $\rho(x) = c_3 x^3 + c_2 x^2 + c_1 x$ with $c_n = k_n \pi^n$, $k_n \in \mathbb{Z}_{>0}$; (A2) flat measure dx ; (A3) $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ (exact); (A4) bulk $\rightarrow$ degree 3, boundary $\rightarrow$ degree 2, edge $\rightarrow$ degree 1. Then ρ in (??) is the unique density of this form, and $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ is uniquely fixed.*

Proof. Given (A3) and (A4), the constraint $c_n/(n+1) = \{4\pi^3, \pi^2, \pi\}$ for $n = \{3, 2, 1\}$ uniquely determines $c_3 = 16\pi^3$, $c_2 = 3\pi^2$, $c_1 = 2\pi$. The integer factors $k_n = (16, 3, 2)$ follow immediately, and (A1) ensures no other polynomial of this form exists. $\square$

This theorem is correct but its hypotheses include CP-2 (A1), CP-3 (A2), and CP-1 (A3). It is a consistency theorem, not a uniqueness proof from S^3 geometry alone.

Failure mode (G1 statement). The derivation of α^{-1} requires the additional free choices of:

- (i) the flat measure dx over the natural S^3 volume measure $x^3 dx$;
- (ii) treating $4\pi^3 + \pi^2 + \pi$ as exact rather than as a 0.22 ppm approximation.

If (i) is dropped, the set of admissible α^{-1} values includes $\mu_3 = 77.06$ and infinitely many others depending on the chosen measure. If (ii) is dropped, the exact theoretical value of α^{-1} is unknown. G1 is false as a theorem of the current TOE axioms, and true only if (i) and (ii) are elevated to axioms.

Definition 4.2 (Minimal closure axiom (G1-fix)). **Axiom G1-fix:** The admissible density on (B^4, S^3) is the unique degree-3 polynomial in $x \in [0, 1]$ under flat measure whose graded integer coefficients are $k_n \in \{2 \cdot \dim \mathbb{O}, \dim(\text{adj } SU(2)), |\mathbb{Z}_2|\}$ for $n = \{3, 2, 1\}$ respectively.

Under G1-fix, $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ follows unconditionally. The axiom is *natural* (each clause references a canonical algebraic object) but *not forced* by the GON dynamics or the operator algebra $\hat{O}$ of Paper 18.

5 Connection to $J_3(\mathbb{O})$

The exceptional Jordan algebra $J_3(\mathbb{O})$ has $\dim_{\mathbb{R}} = 27$ and automorphism group F_4 ($\dim = 52$). Its Peirce decomposition with respect to the three primitive idempotents is

$$J_3(\mathbb{O}) = \bigoplus_{i=1}^3 \mathbb{R} e_i \oplus \bigoplus_{i < j} V_{ij}, \quad V_{ij} \cong \mathbb{O}, \quad (3)$$

giving $3 + 3 \cdot 8 = 27$ dimensions.

Integer pattern. Each diagonal idempotent e_i is adjacent to exactly two off-diagonal sectors (V_{ij} and V_{ik}), of combined dimension $2 \cdot 8 = 16$. The diagonal subspace $\bigoplus \mathbb{R} e_i$ has dimension 3. Each V_{ij} carries a canonical $\mathbb{Z}_2$ grading from the Cayley–Dickson construction $\mathbb{H} \rightarrow \mathbb{O}$, contributing the factor 2. The integer pattern $(16, 3, 2)$ is therefore read off from the Peirce decomposition without circularity or auxiliary choice.

Does $J_3(\mathbb{O})$ fix the measure? The unique F_4 -invariant inner product on $J_3(\mathbb{O})$ is the trace form $\langle A, B \rangle = \text{Tr}(A \circ B)$. If the integration measure $d\mu$ on $[0, 1]$ is required to be F_4 -equivariant with respect to the Peirce grading (i.e. to weight each sector by its F_4 -invariant volume), one expects the measure on the “bulk” radial direction to be flat, since the trace form on the off-diagonal sectors $V_{ij} \cong \mathbb{O}$ is already the standard Euclidean measure on $\mathbb{R}^8$ (no radial warping). This argument is suggestive but not yet a proof.

Proposition 5.1 ($J_3(\mathbb{O})$ as partial constraint). *The Peirce decomposition of $J_3(\mathbb{O})$ determines the integer factors $(16, 3, 2)$ without circularity. It does not independently fix the integration measure or the polynomial functional form. The hypothesis that the F_4 -invariant measure on the radial interval $[0, 1]$ is flat would close CP-3; this hypothesis is plausible from the trace-form structure but requires explicit proof.*

6 Open Items

1. **Resolve CP-1 (exactness).** Either derive the 0.22 ppm shift from first principles (QED corrections to the bare geometric formula) or find the exact expression that equals α_{exp}^{-1} . Until this is done G1 is not well-posed.

2. **Close CP-3 (measure) via $J_3(\mathbb{O})$.** Prove or disprove that the F_4 -invariant volume form on the Peirce-graded radial direction is flat. This is the most tractable near-term target and the most structurally significant gap.
3. **Close CP-2 (functional form) via minimality.** Prove that degree 3 is the minimum polynomial degree compatible with the three-layer Peirce grading and flat measure. (Degree 2 integrates to ≈ 13.01 , far from α^{-1} , so minimality is plausible by inspection.)
4. **Unify CP-4 (edge coefficient).** Adopt the $J_3(\mathbb{O})$ Peirce reading (2 from $\mathbb{Z}_2$ grading of V_{ij}) as canonical and rule out the representation-dimension reading, or prove they are equivalent in this context.
5. **Cross-reference Paper 18 ($\hat{O}$) assemblies.** If the master operator $\hat{O}$ uniquely selects an assembly on (B^4, S^3) , check whether that assembly determines the integration measure. This is the most direct internal route to closing G1 from within the TOE.

Summary. The derivation of $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ is internally consistent and geometrically motivated but is not a uniqueness proof from S^3 geometry alone. Four choice points are identified; the two load-bearing ones are the exactness of the numerical identity (CP-1) and the choice of flat integration measure (CP-3). $J_3(\mathbb{O})$ eliminates the need for any auxiliary choice in deriving the integer factors (16, 3, 2) and provides a plausible route to closing CP-3 via the F_4 -invariant trace form. Axiom G1-fix closes the proof conditionally but is not itself derivable from the current TOE axioms. G1 remains open.