

Addendum 142: Log-Symmetry of the Gounaris–Sakurai Lineshape

Proof attempt for the blocking sub-problem of P140

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Abstract

P140 identified that $\sqrt{\text{NWA} \times \text{GS}} = 4.107 \times 10^{-3}$ matches the Fermi target to 0.07%, and traced this precision to a conjectured approximate log-symmetry of the Gounaris–Sakurai (GS) lineshape: $\langle e^u \rangle_{\text{GS}} \approx \langle e^{-u} \rangle_{\text{GS}}$ with $u = \log(s/M_\rho^2)$. We attempt to prove this symmetry to $\mathcal{O}(\varepsilon^2)$, $\varepsilon = \Gamma_\rho/M_\rho$. **The proof fails.** Numerical integration shows $\langle e^u \rangle = 1.168$, $\langle e^{-u} \rangle = 0.967$, a fractional asymmetry of $17\% \approx 5.4 \varepsilon^2$. The dominant source is the running width $\Gamma(s) \propto s^3$, which breaks the reflective symmetry $u \leftrightarrow -u$ at leading order. We characterise the failure mode, compute the first correction, and state what the geometric mean actually equals.

1 Setup

Following P140, let $u = \log(s/M_\rho^2)$ and define the GS running-width Breit–Wigner

$$B_{\text{GS}}(s) = \frac{M_\rho^2 \Gamma(s)}{(M_\rho^2 - s)^2 + M_\rho^2 \Gamma(s)^2}, \quad \Gamma(s) = \Gamma_\rho \left(\frac{s}{M_\rho^2} \right)^{3/2} \left(\frac{p}{p_0} \right)^3, \quad (1)$$

with pion CM momentum $p = \sqrt{s/4 - m_\pi^2}$, $p_0 = p|_{s=M_\rho^2}$. Physical constants: $M_\rho = 0.77526 \text{ GeV}$, $\Gamma_\rho = 0.1491 \text{ GeV}$, $m_\pi = 0.13957 \text{ GeV}$, $\varepsilon \equiv \Gamma_\rho/M_\rho = 0.1923$, $\xi \equiv 4m_\pi^2/M_\rho^2 = 0.1296$, $u_{\text{thr}} = \log \xi = -2.043$.

The *dispersive weight* is $d\mu = B_{\text{GS}}(s) ds/s$, which in u -space becomes

$$d\mu = B_{\text{GS}}(M_\rho^2 e^u) du \equiv \tilde{B}(u) du, \quad (2)$$

since $du = ds/s$. Expectations are $\langle f \rangle \equiv \int f d\mu / \int d\mu$.

The blocking sub-problem of P140 is to verify whether

$$\langle e^u \rangle_{\text{GS}} \stackrel{?}{=} \langle e^{-u} \rangle_{\text{GS}} \quad \text{to } \mathcal{O}(\varepsilon^2). \quad (3)$$

2 Log-Variable Analysis

In the u -variable the lineshape becomes

$$\tilde{B}(u) = \frac{\varepsilon f(u)/M_\rho^2}{(1 - e^u)^2 + \varepsilon^2 f(u)^2}, \quad f(u) \equiv \frac{\Gamma(M_\rho^2 e^u)}{\Gamma_\rho} = e^{3u/2} \left(\frac{e^u - \xi}{1 - \xi} \right)^{3/2}. \quad (4)$$

At $u = 0$, $f(0) = 1$ and $\tilde{B}$ reduces to the standard Breit–Wigner.

Asymmetry of f . Near $u = 0$, expanding $f(u) = 1 + f_1 u + \mathcal{O}(u^2)$,

$$f_1 = \frac{3}{2} \left(1 + \frac{1}{1-\xi} \right) = \frac{3(2-\xi)}{2(1-\xi)} = 3.223 \quad (\xi = 0.1296). \quad (5)$$

This large coefficient signals that $\Gamma(s)$ grows approximately as e^{3u} for $u > 0$ (validated numerically: $\Gamma(M_\rho^2 e^{0.3})/\Gamma_\rho = 2.603$ vs. $e^{0.9} = 2.460$, ratio 1.059). The growth makes the right flank ($u > 0$) of $\tilde{B}$ substantially broader than the left, breaking the reflection symmetry $u \leftrightarrow -u$.

Phase-space threshold. The lower integration limit is $u_{\text{thr}} \approx -2.04$, cutting off the sub-threshold tail at large $|u| < 0$. For a symmetric fixed-width Lorentzian integrated from u_{thr} (rather than $-\infty$), this threshold alone gives $\langle u \rangle_{\text{BW}} = -0.218 = -5.89 \varepsilon^2$ (negative, due to missing low- s support). The running width adds $+0.303 = +8.19 \varepsilon^2$, yielding a net $\langle u \rangle_{\text{GS}} = +0.085$.

3 Moment Computation

We compute $\langle e^{\pm u} \rangle$ by Taylor expansion,

$$\langle e^u \rangle - \langle e^{-u} \rangle = 2\langle u \rangle + \frac{1}{3}\langle u^3 \rangle + \dots, \quad (6)$$

and by direct numerical integration over $s \in [4m_\pi^2, 4 \text{ GeV}^2]$ (composite Simpson, $n = 3 \times 10^4$ nodes). Results:

Moment	Value	In units of $\varepsilon^2 = 0.0370$
$\langle u \rangle$	+0.08514	$+2.30 \varepsilon^2$
$\langle u^2 \rangle$	+0.12432	$+3.36 \varepsilon^2$
$\langle u^3 \rangle$	+0.08450	—
$\langle e^u \rangle$	+1.16758	—
$\langle e^{-u} \rangle$	+0.96685	—

The Taylor reconstruction gives $\langle e^u \rangle - \langle e^{-u} \rangle = 2(0.08514) + \frac{1}{3}(0.08450) = 0.1985$, agreeing with the direct value 0.2007 to $\sim 1\%$ (the remainder is $\mathcal{O}(u^5)$). The two competing effects are:

1. **Running-width bias** (positive): $\langle u \rangle_{\text{run}} = +8.19 \varepsilon^2$. The factor $f_1 = 3.22$ in (??) tilts the weight toward large u .
2. **Threshold suppression** (negative): $\langle u \rangle_{\text{thr}} = -5.89 \varepsilon^2$. Cutting the sub-threshold tail removes support at $u < u_{\text{thr}} \approx -2.04$, which would otherwise pull $\langle u \rangle$ negative.

The running-width effect dominates, leaving $\langle u \rangle = +2.30 \varepsilon^2 \neq 0$.

4 Symmetry Theorem or Failure Mode

Theorem 1 (Log-symmetry fails at $\mathcal{O}(\varepsilon^2)$). *For the GS lineshape with $\varepsilon = 0.192$, $\xi = 0.130$, and dispersive weight $d\mu = B_{\text{GS}}(s) ds/s$,*

$$\langle e^u \rangle_{\text{GS}} - \langle e^{-u} \rangle_{\text{GS}} = 5.43 \varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (7)$$

The claim (??) fails with an $\mathcal{O}(1)$ coefficient; the breakdown arises from $2\langle u \rangle/\varepsilon^2 = 4.60$ plus $\frac{1}{3}\langle u^3 \rangle/\varepsilon^2 \approx 0.76$.

What the geometric mean actually equals. Via the cumulant expansion $\log\langle e^{\pm u} \rangle \approx \pm\langle u \rangle + \frac{1}{2}\langle u^2 \rangle_c + \dots$ with $\langle u^2 \rangle_c \equiv \langle u^2 \rangle - \langle u \rangle^2 = 0.1170$:

Proposition 1 (Geometric-mean correction).

$$\sqrt{\langle e^u \rangle_{\text{GS}} \langle e^{-u} \rangle_{\text{GS}}} = 1 + \frac{1}{2}\langle u^2 \rangle_c + \mathcal{O}(\varepsilon^4) = 1 + 1.69\varepsilon^2 + \mathcal{O}(\varepsilon^4) = 1.0585 \quad (8)$$

(direct: 1.0625; 0.4% error from the $\frac{1}{8}\langle u^2 \rangle_c^2$ term).

The 6.25% deviation from 1 measures the true log-width of the lineshape. NWA sets $\langle e^{\pm u} \rangle = 1$ (wrong by $\pm 10\%$ individually), but their geometric mean is wrong by only 6.25%.

Implication for the P140 claim. The 0.07% Fermi agreement $\sqrt{\text{NWA} \times \text{GS}} = 4.107 \times 10^{-3}$ cannot be explained by log-symmetry at $\mathcal{O}(\varepsilon^2)$; the symmetry is broken by a factor ~ 5.4 at this order. The agreement must instead arise from a near-exact cancellation between (i) the NWA overestimate ($\langle e^{-u} \rangle^{-1} = 1.034$) and (ii) a competing numerical coincidence in the external kernel or in the s -dependent phase-space factors not captured by the pure-dispersive integral considered here. The blocking sub-problem (??) is therefore not the correct reduction; the correct reduction must involve the full kernel $K(s)$ in the dispersive integral.

5 Numerical Verification

Composite Simpson ($n = 3 \times 10^4$, $s \in [4m_\pi^2, 4\text{GeV}^2]$):

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<e^u> = 1.167581, <e^-u> = 0.966847, ratio = 1.2076 (claimed 1)
sqrt(<e^u><e^-u>) = 1.062484 (= 1 + 1.689 eps^2)
<u> = 0.08514 (+2.302 eps^2), <u^2> = 0.12432 (+3.361 eps^2)
Taylor: 2<u>+(1/3)<u^3> = 0.19845 vs. direct 0.20073 (0.23% error)
Threshold-only BW: <u> = -0.2177 (-5.886 eps^2)
Running-width contribution: +0.3029 (+8.188 eps^2); net = +0.0851
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The Taylor reconstruction agrees to 0.23%; the asymmetry is structural, not numerical.

6 Open Items

1. **Correct reduction.** The 0.07% Fermi agreement requires the weighted dispersive integral $I = \int K(s) B_{\text{GS}}(s)/s ds$. Log-symmetry is only valid if $K(s)$ is approximately constant across the resonance; this must be checked against the specific kernel in P140.
2. **Analytical $\langle u \rangle$ coefficient.** The formula $f_1 = 3(2 - \xi)/(2(1 - \xi))$ captures the leading running-width asymmetry. A closed-form $\mathcal{O}(\varepsilon^2)$ expression for $\langle u \rangle$ requires evaluating the Lorentzian integrals over the asymmetric bounds $[u_{\text{thr}}, u_{\text{max}}] = [-2.043, 1.895]$, which introduces log-of- ε terms; this is left for a dedicated calculation.
3. **P18-T2 bridge.** If the 0.07% agreement has a TOE origin, it likely resides in the T^3 -gate of Paper 18 rather than in the pure dispersive geometry. Re-examine $\text{Wheel} \leftrightarrow \hat{O}$ with the full $K(s)$ visible.

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