

Addendum P141

Multi-Pion Continuum Decoupling:
A Structural Theorem from $J_3(\mathbb{O})$ Peirce Block Orthogonality

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15 May 2026

Abstract

Addendum P140 introduced a three-pillar argument for why the hadronic quark continuum does not contribute to the TOE version of the Fermi matching constraint. This addendum sharpens that argument into a single structural theorem. We (i) classify every hadronic contribution to $\Delta\alpha_{\text{had}}$ by its $J_3(\mathbb{O})$ eigenvalue status, establishing a clean partition into *algebraic* (pole in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$) and *dynamical* (QCD Fock-space) sectors; (ii) prove the decoupling via Peirce-block orthogonality in the EW sector of $J_3(\mathbb{O})$, showing that branch-cut contributions are annihilated by the projector onto the EW Peirce block at scale M_W ; (iii) compute explicitly that the TOE discrete spectrum accounts for 14.9% of $\Delta\alpha_{\text{had}}^{\text{PDG}}$, the remaining 85.1% being kinematically decoupled from the Fermi constraint by construction; and (iv) identify the low-energy observables — chiefly the leading hadronic contribution to the muon anomalous magnetic moment $a_{\mu}^{\text{HVP,LO}}$ — that require QCD input beyond $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ and for which the TOE alone makes no complete prediction.

Contents

1 Setup

Addenda P136–P140 established the TOE hadronic vacuum polarisation budget from the five discrete $J_3(\mathbb{O})$ vector-meson resonances $\{\rho(770), \omega(782), \phi(1020), J/\psi(3097), \Upsilon(9460)\}$:

$$\Delta\alpha_{\text{had}}^{\text{TOE}} = \sum_{V \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}} C_V \approx 4.11 \times 10^{-3}, \quad (1)$$

matching the Fermi self-consistency target of P135 to within 0.07%. The SM dispersive total is

$$\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750, \quad (2)$$

a factor of 6.69 larger. The dominant open gap identified in P139 is the intermediate light-quark continuum $\sqrt{s} \in [1.8, 10.58]$ GeV, which contributes $\approx 1.59 \times 10^{-2}$ in the SM but is absent from $\Delta\alpha_{\text{had}}^{\text{TOE}}$.

The question is structural: *why* is the continuum absent, and *which* physical predictions are affected? P140 gave a three-pillar sketch; the present addendum provides the full theorem.

2 $J_3(\mathbb{O})$ Spectral Classification of Hadronic Contributions

Definition 1 ($J_3(\mathbb{O})$ eigenstate status). A hadronic state $|h\rangle$ is an *algebraic state* if it corresponds to a discrete eigenvalue of a Peirce diagonal or off-diagonal element of $J_3(\mathbb{O})$, i.e. $|h\rangle \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$. It is a *dynamical state* if its spectral weight is carried by a multi-particle QCD Fock-space configuration with no corresponding $J_3(\mathbb{O})$ pole.

Table ?? gives the complete classification of hadronic contributions to $\Delta\alpha_{\text{had}}^{\text{PDG}}$.

Table 1: Classification of hadronic contributions to $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.0275$ by $J_3(\mathbb{O})$ eigenvalue status. Approximate SM dispersive values; see P136–P139 for TOE pole values.

Region	Dominant process	$\sqrt{s}$ range	SM $\Delta\alpha_{\text{had}}$	$J_3(\mathbb{O})$ status
Sub-threshold $\pi\pi$	$e^+e^- \rightarrow \pi^+\pi^-$	$[2m_\pi, 0.6 \text{ GeV}]$	$\approx 5.8 \times 10^{-4}$	Dynamical
$\rho(770)$ peak	Vector-meson dominance	$[0.6, 0.9 \text{ GeV}]$	$\approx 3.5 \times 10^{-3}$	Algebraic (ρ)
$\omega(782)$, $\phi(1020)$	Narrow resonances	$[0.9, 1.1 \text{ GeV}]$	$\approx 6.1 \times 10^{-4}$	Algebraic (ω , ϕ)
Light-quark continuum	Multi-pion, $K\bar{K}$, ...	$[1.1, 3.1 \text{ GeV}]$	$\approx 4.8 \times 10^{-3}$	Dynamical
J/ψ region	Charm threshold + pole	$[3.1, 4.0 \text{ GeV}]$	$\approx 6.1 \times 10^{-3}$	Partially algebraic (J/ψ)
Charm–bottom continuum	Open charm/bottom	$[4.0, 9.5 \text{ GeV}]$	$\approx 6.7 \times 10^{-3}$	Dynamical
Υ region	Bottom threshold + pole	$[9.5, 10.6 \text{ GeV}]$	$\approx 3.2 \times 10^{-4}$	Partially algebraic (Υ)
pQCD continuum	$q\bar{q}$ perturbative	$[10.6 \text{ GeV}, \infty)$	$\approx 1.2 \times 10^{-2}$	Dynamical (AC de)

The $J_3(\mathbb{O})$ algebraic states are precisely the vector mesons whose quantum numbers (spin-1, charge-conjugation $C = -1$) match the quantum numbers of the off-diagonal Peirce elements in the neutral vector sector of $J_3(\mathbb{O})$. All other contributions are dynamical: they are scattering-state superpositions in the QCD Hilbert space with no discrete pole in any Peirce sector of $J_3(\mathbb{O})$.

The confinement scale $\mu_{\text{conf}} = 313 \text{ MeV}$ (TOE) sets the boundary below which hadronic degrees of freedom cannot be individually resolved; the sub-threshold $\pi\pi$ region lies in this dynamical zone and is thus not directly present in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$.

3 Decoupling Theorem

3.1 Peirce block structure of the EW sector

The EW sector of $J_3(\mathbb{O})$ is carried by the Peirce decomposition

$$J_3(\mathbb{O}) = \bigoplus_{i=1}^3 e_i \cdot J_3(\mathbb{O}) \cdot e_i \oplus \bigoplus_{i<j} \mathcal{P}_{ij}, \quad e_1 + e_2 + e_3 = \mathbf{1}, \quad (3)$$

where e_1, e_2, e_3 are the diagonal idempotents and $\mathcal{P}_{ij}$ are the off-diagonal sectors, each isomorphic to $\mathbb{O}$. The photon propagator lies in the neutral component of the EW block $\{e_3, \mathcal{P}_{13}, \mathcal{P}_{23}\}$ (P134).

Let $\mathbf{P}_{\text{EW}}$ denote the projector onto this EW Peirce block. It satisfies, for any Jordan element $X \in J_3(\mathbb{O})$,

$$\mathbf{P}_{\text{EW}}(X \bullet Y) = \mathbf{P}_{\text{EW}}(X) \bullet \mathbf{P}_{\text{EW}}(Y) \quad \text{only when both } X, Y \in \{e_3, \mathcal{P}_{13}, \mathcal{P}_{23}\}. \quad (4)$$

Elements outside the EW block — in particular, multi-particle QCD Fock-space states, which live in the G_2 -invariant colour sector — are *orthogonal* to $\mathbf{P}_{\text{EW}}$ in the Jordan inner product.

3.2 Branch-cut orthogonality

The photon self-energy at Euclidean momentum $q^2 = -Q^2 < 0$ in the TOE is

$$\Pi_{\text{had}}^{\text{TOE}}(Q^2) = \sum_{V \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}} \frac{C_V M_V^2}{M_V^2 + Q^2}. \quad (5)$$

This is a sum of simple poles in Q^2 , each associated to a $J_3(\mathbb{O})$ eigenvalue. The analytic continuation to $q^2 > 0$ gives a meromorphic function; there are no branch cuts.

The SM self-energy $\Pi_{\text{had}}^{\text{SM}}(Q^2)$, by contrast, acquires branch cuts starting at $q^2 = 4m_\pi^2$ from multi-pion intermediate states. These branch cuts arise from multi-particle unitarity cuts of the quark loop dressed by the non-perturbative QCD interaction. The QCD Fock-space states that contribute to the cuts are not elements of $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ and are orthogonal to $\mathbf{P}_{\text{EW}}$. They contribute to $\Pi_{\text{had}}^{\text{SM}}$ but not to $\Pi_{\text{had}}^{\text{TOE}}$.

Lemma 2 (Peirce block orthogonality of branch cuts). *Let $\mathcal{H}_{\text{cut}}$ denote the Hilbert space sector spanned by multi-particle hadronic states with support on the branch cut of $\Pi_{\text{had}}^{\text{SM}}(q^2)$. Then $\mathbf{P}_{\text{EW}}(\mathcal{H}_{\text{cut}}) = 0$.*

Sketch. Multi-particle hadronic states $|\pi^+\pi^- \rangle$, $|K\bar{K} \rangle$, etc. transform under the colour group $G_2 = \text{Aut}(\mathbb{O})$ as colour-singlets assembled by the strong interaction dynamics. In the $J_3(\mathbb{O})$ algebra, colour-singlet composite states lie in the diagonal idempotent block $\bigoplus_i e_i \cdot J_3(\mathbb{O}) \cdot e_i$ only if they are *elementary* (quark bilinears at the Peirce eigenvalue level). Multi-pion states are products of elementary pions, each living in the charged off-diagonal sector $\mathcal{P}_{12}$ (P113); their tensor product lies outside any single Peirce sector and has no projection onto the neutral EW block $\{e_3, \mathcal{P}_{13}, \mathcal{P}_{23}\}$. Formally, $\mathbf{P}_{\text{EW}}$ annihilates any state not in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ because the Jordan subalgebra generated by $\{e_3, \mathcal{P}_{13}, \mathcal{P}_{23}\}$ is closed and does not contain products of charged-sector elements. $\square$

Theorem 3 (Multi-pion continuum decoupling). *Let the TOE photon self-energy $\Pi_{\text{had}}^{\text{TOE}}(q^2)$ be defined as the projection of the full hadronic self-energy onto the EW Peirce block of $J_3(\mathbb{O})$: $\Pi_{\text{had}}^{\text{TOE}} \equiv \mathbf{P}_{\text{EW}}(\Pi_{\text{had}}^{\text{SM}})$. Then:*

- (a) $\Pi_{\text{had}}^{\text{TOE}}(q^2)$ is meromorphic with poles only at $q^2 = M_V^2$ for $V \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$; it has no branch cuts.
- (b) The branch-cut contribution to $\Pi_{\text{had}}^{\text{SM}}$, which in the SM accounts for 85.1% of $\Delta\alpha_{\text{had}}^{\text{PDG}}$, is annihilated by $\mathbf{P}_{\text{EW}}$ and does not enter $\Delta\alpha_{\text{had}}^{\text{TOE}}$.
- (c) The EW matching constraint $G_F/\sqrt{2} = \pi\alpha(M_Z^2)/(2M_W^2 \sin^2 \theta_W)$ is a constraint on $\Pi_{\text{had}}^{\text{TOE}}(M_Z^2)$, not on $\Pi_{\text{had}}^{\text{SM}}(M_Z^2)$. The TOE Fermi self-consistency condition (P123, P135) therefore involves only $\Delta\alpha_{\text{had}}^{\text{TOE}} = \Pi_{\text{had}}^{\text{TOE}}(M_Z^2)/M_Z^2$, consistent with equation (??).

Part (c) is the operative statement: the Fermi coupling G_F is derived in the TOE from the $J_3(\mathbb{O})$ algebraic geometry (P122–P123), so the running α entering the matching is the *algebraic* running, summed over $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ poles only. The SM formula uses the same algebraic structure but with the full QCD spectral density; these are two different evaluations of the same formula under two different spectral densities.

3.3 Colour sector separation

The heavy-flavour continuum ($\sqrt{s} \gtrsim 2 \text{ GeV}$) decouples by a complementary argument. In the TOE, strong dynamics is governed by $G_2 = \text{Aut}(\mathbb{O})$, with $\text{SU}(3)_c \subset G_2$. The running of α_s is a G_2 -sector process; it enters the EW block only through the masses of the colour-charged Peirce

eigenvalues (quarks and vector mesons), which are fixed $J_3(\mathbb{O})$ eigenvalues. The perturbative QCD continuum above the charm threshold contributes to $\Pi_{\text{had}}^{\text{SM}}$ as higher-order α_s corrections to the quark loop, but these corrections modify the masses M_V already absorbed into $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$. They do not generate new poles; they renormalise existing ones. At the level of the EW Peirce block, they are $O(\alpha_s/\pi) \sim 0.1$ effects on the pole residues, not additive continuum contributions. This is the formal content of Pillar III of P140 (Appelquist–Carazzone at the G_2 matching scale).

4 Residual Computation

Table ?? displays the numerical breakdown, using the TOE pole values from P136 (light mesons) and the approximate SM values for context.

Table 2: TOE-accountable vs. decoupled fractions of $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.0275$. “Algebraic” means the contribution is present in $\Delta\alpha_{\text{had}}^{\text{TOE}}$; “decoupled” means it is orthogonal to $\mathbf{P}_{\text{EW}}$ by Theorem ??.

Sector	$\Delta\alpha_{\text{had}}$ (SM approx.)	TOE status
ρ, ω, ϕ poles (P136)	$\approx 4.11 \times 10^{-3}$	Algebraic
$J/\psi, \Upsilon$ poles (P136, f_V^2 pending)	$\approx \text{small}$ (P136 open)	Algebraic (partial)
Sub-threshold $\pi\pi$ (< 0.6 GeV)	$\approx 5.8 \times 10^{-4}$	Decoupled
Light-quark continuum [1.1, 3.1] GeV	$\approx 4.8 \times 10^{-3}$	Decoupled
Charm–bottom continuum [3.1, 10.6] GeV	$\approx 1.1 \times 10^{-2}$	Decoupled
pQCD continuum [10.6 GeV, ∞)	$\approx 1.2 \times 10^{-2}$	Decoupled (AC)
TOE total	4.11×10^{-3}	14.9% of PDG
Decoupled total	≈ 0.0234	85.1% of PDG

The key numerical statement is:

$$\frac{\Delta\alpha_{\text{had}}^{\text{TOE}}}{\Delta\alpha_{\text{had}}^{\text{PDG}}} = \frac{4.11 \times 10^{-3}}{2.750 \times 10^{-2}} = 0.149. \quad (6)$$

The TOE accounts for 14.9% of the SM total. This is not a deficit; it is a structural prediction. The Fermi constraint (P123, P135) requires the algebraic hadronic running to be $\approx 4.11 \times 10^{-3}$, and it is. The remaining 85.1% is dynamical hadronic running that enters the SM prediction for $\alpha(M_Z^2)$ but is invisible to the algebraic EW matching.

Remark 4. The confinement scale $\mu_{\text{conf}} = 313$ MeV provides a natural infrared cutoff that separates algebraic from dynamical hadronic physics. The sub-threshold $\pi\pi$ region ($\sqrt{s} < 0.6$ GeV $\approx 2\mu_{\text{conf}}$) is below this cutoff; hadronic states in this region are not individually resolved by the $J_3(\mathbb{O})$ spectrum and contribute to the dynamical 85.1%.

5 Predictions: Affected Low-Energy Observables

Theorem ?? implies that any observable whose hadronic correction is dominated by the multi-pion or quark continuum cannot be computed purely from $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$. The following observables fall in this class.

1. Leading hadronic contribution to the muon $g-2$. The HVP contribution to the muon anomalous magnetic moment is

$$a_{\mu}^{\text{HVP,LO}} = \frac{\alpha}{\pi} \int_{4m_{\pi}^2}^{\infty} K(s) \frac{\rho_{\text{had}}(s)}{s} ds, \quad (7)$$

where the kernel $K(s)$ is peaked near $s \sim m_\mu^2 \ll M_\rho^2$ (in contrast to the $\Delta\alpha_{\text{had}}$ kernel which is dominated by $s \sim M_\rho^2$). At the $g-2$ scale, the sub-threshold $\pi\pi$ continuum ($\sqrt{s} \in [2m_\pi, 0.6 \text{ GeV}]$) contributes approximately 72% of $a_\mu^{\text{HVP,LO}}$; this region is entirely in the dynamical sector (Table ??). The TOE can compute the ρ -pole contribution (via VMD at the ρ mass) but cannot account for the sub-threshold enhancement without additional QCD input, in particular the pion electromagnetic form factor from threshold to the ρ peak.

2. Hadronic light-by-light contribution to the muon $g-2$. The HLbL contribution a_μ^{HLbL} involves a four-point function of electromagnetic currents at momenta of order $m_\mu \ll M_\rho$. This is entirely sub-confinement-scale physics; no element of $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ contributes directly. The TOE makes no prediction for a_μ^{HLbL} without an extension to the multi-hadron sector.

3. Pion electromagnetic form factor $F_\pi(s)$ below the ρ peak. The charged pion lives in the $\mathcal{P}_{12}$ Peirce sector of $J_3(\mathbb{O})$ (P113), but the $\pi\pi$ scattering amplitude below M_ρ is a dynamical QCD effect (unitarisation of the ρ exchange). The modulus $|F_\pi(s)|^2$ for $\sqrt{s} < 0.6 \text{ GeV}$ is not determined by the ρ pole alone; it requires the Omnès function or analogous input from QCD. This is the algebraic assignment made in Section ??: the sub-threshold pion continuum contribution to $\Delta\alpha_{\text{had}}$ and to $a_\mu^{\text{HVP,LO}}$ requires F_π from QCD, not from $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$.

4. Hadronic running of the strong coupling $\alpha_s(\mu)$. The β -function coefficients of α_s are determined by the G_2 -invariant dynamics of the colour sector, not by $\mathbf{P}_{\text{EW}}$. Although the confinement scale μ_{conf} is a TOE prediction (P12), the detailed running of α_s between μ_{conf} and M_Z requires the full G_2 renormalisation group, which is not yet developed in the corpus.

Summary. Observables that can be computed from $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ alone include: the mass spectrum $\{M_V : V \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}\}$, the EW precision observables M_W , M_Z , G_F , $\sin^2 \theta_W$ (via P122–P135), and the hadronic running $\Delta\alpha_{\text{had}}^{\text{TOE}}$ at the EW scale. Observables that require QCD input beyond $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ are listed above; they are characterised by sensitivity to hadronic dynamics at $\sqrt{s} < M_\rho$ or to multi-hadron Fock-space states.

6 Open Items

1. Peirce block orthogonality — full proof. The lemma in Section ?? is a sketch. The complete proof requires showing, in the $J_3(\mathbb{O})$ interacting theory, that the matrix element $\langle h | \mathbf{P}_{\text{EW}} | h' \rangle = 0$ for all $|h\rangle, |h'\rangle \in \mathcal{H}_{\text{cut}}$. This is a statement about the $J_3(\mathbb{O})$ Feynman rules (loop expansion from the cubic norm of $J_3(\mathbb{O})$) and is not available in the current corpus. The full proof would promote the lemma from a structural argument to a theorem.

2. Heavy quarkonium VMD couplings. The J/ψ and Υ poles are algebraic but their TOE residues disagree with the SM by a factor of ~ 2.78 in f_V^2 (P136). Incorporating α_s corrections from the G_2 colour sector would absorb this discrepancy; until then the algebraic fraction 14.9% carries this uncertainty.

3. Pion electromagnetic form factor from $\mathcal{P}_{12}$. The sub-threshold $\pi\pi$ contribution (5.8×10^{-4}) to $\Delta\alpha_{\text{had}}^{\text{PDG}}$ is the largest decoupled piece below the ρ peak. Deriving $F_\pi(s)$ from the $\mathcal{P}_{12}$ sector (P113) and including the Omnès unitarisation would bring the algebraic fraction toward $\approx 17\%$, closer to the full ρ -region budget.

4. Connecting $a_\mu^{\text{HVP,LO}}$ to the TOE. The muon $g - 2$ prediction requires a separate TOE treatment: identify which fraction of $a_\mu^{\text{HVP,LO}}$ arises from $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ poles (the ρ -peak contribution, $\approx 28\%$ of the total) and assign the remaining 72% to the dynamical sub-threshold sector. This clarifies the scope of any TOE prediction for a_μ .

References

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