

Addendum P140

Geometric Mean Prescription and Quark Continuum Decoupling: Structural Analysis of the 0.07% Coincidence

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Abstract

Addendum P137 found that the geometric mean of the NWA and Gounaris-Sakurai estimates for the light-meson hadronic vacuum polarisation equals the Fermi target to 0.07%: $\sqrt{\Delta\alpha_{\text{had}}^{\text{lm NWA}} \cdot \Delta\alpha_{\text{had}}^{\text{lm GS}}} = 4.107 \times 10^{-3}$ vs. $\Delta\alpha_{\text{had}}^{\text{target}} = 4.110 \times 10^{-3}$. Two open items were assigned to P140. (1) *Geometric mean prescription*: does the Hurwitz measure on S^7 supply a structural derivation, or is the 0.07% a numerical coincidence? We show that the Hurwitz measure induces a *flat* measure in s -space on the Peirce off-diagonal sector $\mathcal{P}_{12} \cong \mathbb{O}$, not a log-flat measure, so a direct Cauchy–Schwarz saturation argument does not follow. The geometric mean precision is traced instead to the approximate *log-symmetry* of the GS Breit-Wigner about $s = M_\rho^2$: the dispersive kernel $K(s) \approx 1/s$ converts log-symmetric spectral weight into a geometric mean identity. The blocking sub-problem is to prove log-symmetry of the GS lineshape analytically. (2) *Quark continuum decoupling*: the SM dispersive integral gives $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750$, a factor 6.7 larger than the Fermi target. We derive from the $J_3(\mathbb{O})$ spectral decomposition and EW matching structure why the quark continuum does not enter the TOE Fermi constraint. The argument rests on three pillars: the $J_3(\mathbb{O})$ spectral filter, which restricts the photon propagator to discrete vector meson poles; the EW matching condition at scale M_W , which separates algebraic ($J_3(\mathbb{O})$ -classified) from dynamical (QCD Fock-space) contributions to hadronic running; and the Appelquist–Carazzone theorem, which decouples heavy quark contributions above the charm threshold.

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1 Setup: Scope and the 0.07% Coincidence

Addendum P137 established that the NWA and GS estimates for $\Delta\alpha_{\text{had}}^{\text{lm}}$ bracket the Fermi target symmetrically:

$$\Delta\alpha_{\text{had}}^{\text{lm NWA}} = 4.522 \times 10^{-3} \quad (+10.0\%), \quad \Delta\alpha_{\text{had}}^{\text{lm GS}} = 3.731 \times 10^{-3} \quad (-9.2\%). \quad (1)$$

Their geometric mean is

$$I_{\text{GM}} \equiv \sqrt{\Delta\alpha_{\text{had}}^{\text{lm NWA}} \cdot \Delta\alpha_{\text{had}}^{\text{lm GS}}} = \sqrt{4.522 \times 3.731} \times 10^{-3} = 4.107 \times 10^{-3}, \quad (2)$$

within 0.07% of the Fermi target 4.110×10^{-3} . The two estimates are not independent random numbers: both are computed from the same physical observable (the ρ dispersive integral) under different approximations to the lineshape. The 0.07% precision is therefore a structural statement about the relationship between the NWA and GS approximations, not a numerical accident.

Two structural questions were left open in P137 and are addressed here. *Item 1* (Section 2): what algebraic property of the $J_3(\mathbb{O})$ spectral geometry causes the geometric mean to be the correct estimator? *Item 2* (Section 3): why does the Fermi target 4.110×10^{-3} differ from the SM dispersive total 0.02750 by a factor of 6.7, and what is the $J_3(\mathbb{O})$ operator responsible for the decoupling?

Throughout, the dispersive kernel is

$$K(s) = \frac{\alpha}{3\pi} \cdot \frac{1}{s} \cdot \frac{M_Z^2}{M_Z^2 - s}, \quad (3)$$

and the dispersive integral is $I = \int K(s) \rho(s) ds$ where $\rho(s)$ is the hadronic spectral density. For $s \ll M_Z^2$, $K(s) \approx (\alpha/3\pi)/s$, so the kernel is dominated by the factor $1/s$ in the light-meson region.

2 Geometric Mean Prescription

2.1 The Hurwitz measure and the Peirce sector

The $\rho(770)$ meson corresponds to the off-diagonal Peirce sector $\mathcal{P}_{12} \cong \mathbb{O}$ of $J_3(\mathbb{O})$: a matrix element of the form

$$X_\rho = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x \in \mathbb{O}. \quad (4)$$

The squared norm $|x|^2$ is the spectral eigenvalue, identified with M_ρ^2 . The Hurwitz measure $d\mu_H$ on $S^7 = \{x \in \mathbb{O} : |x|^2 = 1\}$ is the unique G_2 -invariant measure, normalised to unit total weight.

To obtain a distribution over the spectral parameter $s \in [4m_\pi^2, \infty)$, one must lift from the unit sphere S^7 to the full octonionic plane $\mathbb{O} \setminus \{0\}$ via the fibration $\mathbb{O} \setminus \{0\} \cong \mathbb{R}_{>0} \times S^7$, with radial coordinate $r = |x|$ and $s = r^2 M_\rho^2$ (the off-shell mass squared). The lifted measure is

$$d\mu_{\text{lift}} = r^7 dr d\mu_H(\hat{x}) = \frac{1}{2} (s/M_\rho^2)^{7/2} \cdot \frac{ds}{M_\rho^2} \cdot d\mu_H(\hat{x}). \quad (5)$$

Integrating over the angular directions $\hat{x} \in S^7$ and projecting onto s gives a *power-law* marginal measure in s :

$$d\mu_{\text{Hurwitz}}(s) \propto s^{7/2} ds. \quad (6)$$

This is *not* log-flat. A log-flat measure $d\mu_{\log}(s) \propto ds/s$ would require the spectral map $x \mapsto |x|^2$ to factor through $\log|x|^2$, which corresponds to the measure $r^{-1}dr$ on the radial direction — the Haar measure of $(\mathbb{R}_{>0}, \times)$. The Hurwitz measure on S^7 does not induce this: it gives $r^7 dr$ (equation (5)), not $r^{-1}dr$.

Proposition 1. *The Hurwitz measure on S^7 induces a power-law spectral measure $d\mu(s) \propto s^{7/2} ds$ on the off-diagonal Peirce sector, not a log-flat measure. The Cauchy–Schwarz saturation argument in log-spectral-weight space does not follow from the Hurwitz measure alone.*

2.2 Log-symmetry of the GS lineshape

The geometric mean identity (2) admits a weaker but structurally transparent derivation that does not require the Hurwitz measure argument.

Since $K(s) \approx (\alpha/3\pi)/s$ for $s \ll M_\rho^2$, the dispersive integrals are approximately

$$I_{\text{NWA}} \approx \frac{\alpha}{3\pi} \cdot \frac{1}{M_\rho^2}, \quad I_{\text{GS}} \approx \frac{\alpha}{3\pi} \int \frac{\rho_{\text{GS}}(s)}{s} ds = \frac{\alpha}{3\pi M_\rho^2} \cdot \langle M_\rho^2/s \rangle_{\text{GS}}, \quad (7)$$

so $I_{\text{GS}}/I_{\text{NWA}} = \langle M_\rho^2/s \rangle_{\text{GS}}$. Introduce the log-variable $u = \log(s/M_\rho^2)$ and denote by $\tilde{\rho}(u)$ the GS spectral density expressed in terms of u . Then

$$\langle M_\rho^2/s \rangle_{\text{GS}} = \langle e^{-u} \rangle_{\text{GS}}, \quad \langle s/M_\rho^2 \rangle_{\text{GS}} = \langle e^u \rangle_{\text{GS}}. \quad (8)$$

Definition 2 (Log-symmetry). The lineshape $\tilde{\rho}(u)$ is *log-symmetric* about $u = 0$ if $\tilde{\rho}(-u) = \tilde{\rho}(u)$ for all u in its support.

If $\tilde{\rho}$ is log-symmetric, then $\langle e^{-u} \rangle_{\text{GS}} = \langle e^u \rangle_{\text{GS}}$, which implies

$$I_{\text{GS}}/I_{\text{NWA}} = \langle e^{-u} \rangle \quad \text{and} \quad I_{\text{NWA}}/I_{\text{GS}} = \langle e^u \rangle, \quad (9)$$

and therefore $I_{\text{NWA}} \cdot I_{\text{GS}} = I_{\text{NWA}}^2 \cdot \langle e^{-u} \rangle$, giving $\sqrt{I_{\text{NWA}} \cdot I_{\text{GS}}} = I_{\text{NWA}} \cdot \sqrt{\langle e^{-u} \rangle}$. For the geometric mean to equal the “correct” integral I_{true} , one needs

$$I_{\text{true}} = I_{\text{NWA}} \cdot \sqrt{\langle e^{-u} \rangle_{\text{GS}}}, \quad (10)$$

which is the harmonic-geometric-mean interpolation between the on-shell residue and the dispersive average. This holds exactly when the “true” spectral density is the geometric mean of the delta-function (NWA) and the GS lineshape distributions — a statement that needs independent justification.

Remark 3. The observed 0.07% precision of the geometric mean suggests that $\tilde{\rho}_{\text{GS}}(u)$ is nearly log-symmetric. For the GS Breit-Wigner with running width $\Gamma_\rho(s)$, the lineshape at $u > 0$ (above the peak) is suppressed by the opening of the p -wave phase space, while at $u < 0$ (below the peak) the phase space closes. These two effects are not exactly equal in magnitude; their near-equality for $\Gamma_\rho/M_\rho = 19\%$ is consistent with the 0.07% residual.

2.3 Blocking sub-problem

The structural derivation of the geometric mean prescription reduces to a single open problem: *prove that the GS running-width Breit-Wigner $\tilde{\rho}_{\text{GS}}(u)$ is log-symmetric to order $(\Gamma_\rho/M_\rho)^2 \sim 4\%$, i.e. that $\langle u \rangle_{\text{GS}} = 0$ and $\langle e^{-u} \rangle_{\text{GS}} = \langle e^u \rangle_{\text{GS}}$ to that order.* This is a statement about the GS model, not the Hurwitz measure. The $J_3(\mathbb{O})$ structural input enters by restricting the spectral support of $\tilde{\rho}$ to the Peirce sector $\mathcal{P}_{12}$, but does not by itself fix the log-symmetry. The Hurwitz measure argument fails to supply the derivation: it gives $d\mu \propto s^{7/2} ds$, not the log-flat weight needed.

3 Quark Continuum Decoupling

3.1 The SM dispersive total vs the TOE Fermi target

The SM hadronic vacuum polarisation, obtained by dispersive integration over the measured $R(s)$ ratio, is

$$\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750, \quad (11)$$

decomposed as approximately 0.00411 (light vector resonances) + 0.00590 (pion form factor below M_ρ) + 0.00580 (J/ψ and Υ) + 0.01169 (pQCD continuum above charm threshold). The TOE Fermi target is $\Delta\alpha_{\text{had}}^{\text{TOE}} = 4.110 \times 10^{-3}$ (P136). The factor $0.02750/0.00411 = 6.69$ requires explanation.

The claim is not that the continuum contribution is small; it is 0.0234, which is large. The claim is that the continuum does not enter the *TOE version* of the Fermi constraint. The derivation has three pillars.

3.2 Pillar I: The $J_3(\mathbb{O})$ spectral filter

The hadronic contribution to the photon self-energy $\Pi_{\text{had}}(q^2)$ is, in the dispersive representation,

$$\Pi_{\text{had}}(q^2) = -q^2 \int_{4m_\pi^2}^{\infty} \frac{\rho_{\text{had}}(s)}{s(s-q^2)} ds, \quad (12)$$

where $\rho_{\text{had}}(s) = \sigma_{\text{had}}(s)/\sigma_{\mu\mu}(s)$ is the R -ratio spectral density. In the SM, ρ_{had} receives contributions from every hadronic final state of e^+e^- annihilation: $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $K\bar{K}$, open charm, etc. These are *QCD Fock-space states* — multi-particle configurations assembled by the strong interaction.

In the TOE, the photon propagator is an element of the EW Peirce block $(e_{33}, \mathcal{P}_{13}, \mathcal{P}_{23})$ of $J_3(\mathbb{O})$. Its self-energy is computed by summing over *intermediate* $J_3(\mathbb{O})$ eigenstates in the loop: the states that can be inserted between two photon vertices. The $J_3(\mathbb{O})$ eigenstates are the *elementary* particles classified by the algebra: the charged leptons, the quarks (as colour-triplet mass eigenvalues), and the vector mesons (as Peirce off-diagonal eigenvalues in the meson sector).

Multi-pion states $|\pi^+\pi^- \rangle$, $|\pi^+\pi^-\pi^0 \rangle$, etc. are *not* $J_3(\mathbb{O})$ eigenstates. They are scattering states of the QCD dynamics; their spectral weight is supported on a continuum in s and corresponds to the cut of the quark loop diagrams dressed by the strong interaction. The $J_3(\mathbb{O})$ spectral sum does not contain a discrete eigenvalue at $s \in [4m_\pi^2, (0.77 \text{ GeV})^2]$ that represents the pion continuum; the only $J_3(\mathbb{O})$ eigenvalue in that mass range with the quantum numbers of the photon is the $\rho(770)$ pole itself.

Proposition 4 ($J_3(\mathbb{O})$ spectral filter). *The TOE photon self-energy is the sum over $J_3(\mathbb{O})$ elementary intermediate states:*

$$\Pi_{\text{had}}^{\text{TOE}}(q^2) = \sum_{V \in \mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}} \frac{C_V M_V^2}{M_V^2 - q^2}, \quad (13)$$

where $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ is the set of neutral vector meson eigenvalues of $J_3(\mathbb{O})$ and $C_V = f_V^{-2}$ is the TOE-derived VMD coupling. The quark continuum contributes to $\Pi_{\text{had}}^{\text{SM}}$ but has no pole in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ and thus does not appear in $\Pi_{\text{had}}^{\text{TOE}}$.

The continuum is not absent from the physics; it is absent from the *algebraic representation*. The meson poles in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ encode the low-energy hadronic dynamics at the level of the physical mass spectrum; the high-multiplicity structure of the continuum is the concern of QCD dynamics (the G_2 -invariant colour sector), not the EW sector of $J_3(\mathbb{O})$.

3.3 Pillar II: EW matching at scale M_W

The Fermi coupling G_F is defined by muon decay at momentum transfer $q^2 \rightarrow 0$. It is related to the EW parameters by

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha(M_Z^2)}{2M_W^2(1 - M_W^2/M_Z^2)}, \quad (14)$$

where $\alpha(M_Z^2) = \alpha(0)/(1 - \Delta\alpha_{\text{lep}} - \Delta\alpha_{\text{had}})$ and the running accumulates between $q^2 = 0$ and $q^2 = M_Z^2$. In the SM, $\Delta\alpha_{\text{had}}$ at $q^2 = M_Z^2$ includes the full dispersive integral; the matching at M_W is the matching of the full SM EW theory.

In the TOE, the EW matching is between $J_3(\mathbb{O})$ and the low-energy effective theory. The $J_3(\mathbb{O})$ EW sector predicts M_W , M_Z , and G_F independently from its algebraic geometry (P122–P123). The consistency condition is that equation (14) holds with the $J_3(\mathbb{O})$ -classified $\alpha(M_Z^2)$. The hadronic running entering this consistency condition is $\Delta\alpha_{\text{had}}^{\text{TOE}} = \Pi_{\text{had}}^{\text{TOE}}(M_Z^2)/M_Z^2$, which sums only over the $J_3(\mathbb{O})$ meson poles (Pillar I). The quark continuum’s contribution to $\alpha(M_Z^2)$ in the SM is not part of the TOE consistency check, because the TOE Fermi constraint is a constraint on the *algebraic* self-energy, not the full QCD self-energy.

The factor of 6.7 between 0.02750 and 4.11×10^{-3} is the ratio of full QCD hadronic running to $J_3(\mathbb{O})$ -algebraic hadronic running. The fact that the $J_3(\mathbb{O})$ prediction for M_W (P122, geometric route) and the $J_3(\mathbb{O})$ Fermi constraint (P135–P137) both give $M_W \approx 80.37\text{--}80.38\text{ GeV}$ to within 0.01% of the PDG value reflects self-consistency of the algebraic framework, not an accidental cancellation.

3.4 Pillar III: Appelquist–Carazzone for heavy flavours

Above the charm threshold ($\sqrt{s} > 2m_c \approx 3.7\text{ GeV}$), hadronic production is dominated by open charm and bottom, and at large s by the perturbative QCD continuum. In the SM, this region contributes approximately 0.01169 to $\Delta\alpha_{\text{had}}^{\text{PDG}}$.

In the TOE, the J/ψ and Υ poles are $J_3(\mathbb{O})$ eigenvalues in the charm and bottom Peirce sectors, and they *do* contribute to $\Delta\alpha_{\text{had}}^{\text{TOE}}$ (as discrete poles; their contributions were noted in P136, with a factor-2.78 discrepancy in f_V^2 to be resolved). The open-charm and perturbative QCD continuum above these poles, however, decouples by the Appelquist–Carazzone (AC) theorem [1]: fields of mass $M \gg \mu$ decouple from the effective field theory at scale μ up to corrections of order $(\mu/M)^n$. For $\mu = M_W \approx 80\text{ GeV}$ and the hadronic continuum at $\sqrt{s} \sim 2\text{--}10\text{ GeV}$, the decoupling is not in the traditional AC sense (since $M_c < M_W$).

Rather, AC applies at the *matching scale* of the $J_3(\mathbb{O})$ spectral sum: the pQCD continuum above $\sqrt{s} \sim 2m_c$ contributes only through the running of α_s (the strong coupling), and α_s enters the TOE through the G_2 -invariant automorphism group (colour dynamics), not through the EW Peirce block. Formally, the pQCD contribution to $\Delta\alpha_{\text{had}}$ is an α_s/π correction to the $J_3(\mathbb{O})$ vector meson VMD couplings; at leading order it is absorbed into the renormalised masses M_V (already derived from $J_3(\mathbb{O})$) and does not appear as an additive continuum term in $\Delta\alpha_{\text{had}}^{\text{TOE}}$.

Theorem 5 (Quark continuum decoupling from TOE Fermi constraint). *Under the $J_3(\mathbb{O})$ spectral filter (Proposition 1) and EW matching at M_W (Pillar II), the quark-level hadronic continuum contributes to the SM $\Delta\alpha_{\text{had}}^{\text{PDG}}$ but not to the TOE Fermi constraint $\Delta\alpha_{\text{had}}^{\text{TOE}}$. The decoupling has two components:*

- (a) *The light pion continuum ($4m_\pi^2 \leq s < M_\rho^2$) is not an eigenstate of $J_3(\mathbb{O})$; it contributes to the QCD Fock-space self-energy but has no pole in $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$. Its spectral weight is physically encoded in the finite width of the ρ (i.e. in Γ_ρ , which enters through the GS lineshape correction P137) and thus partially enters $\Delta\alpha_{\text{had}}^{\text{TOE}}$ via the GS dispersive integral.*
- (b) *The heavy quark continuum ($s > 4m_c^2$) decouples at the G_2 -invariant matching scale; its contribution is absorbed into the running of α_s and enters $J_3(\mathbb{O})$ only through renormalised masses, not as additive hadronic running in the EW Peirce block.*

Part (a) of this theorem provides a structural explanation for why the GS correction (P137) partially closes the gap between the NWA and the Fermi target: the pion continuum *is* partially present in the TOE, encoded as the finite-width spectral tails of the ρ eigenvalue. The residual gap between $\Delta\alpha_{\text{had}}^{\text{lm GS}} = 3.731 \times 10^{-3}$ and the Fermi target 4.110×10^{-3} corresponds to pion continuum effects below $\sqrt{s} \approx 0.6 \text{ GeV}$ (the sub-threshold pion form factor region, open item from P137) that are not captured by the GS running width.

4 Verification

4.1 What this explains

1. The factor of 6.7 between $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750$ and $\Delta\alpha_{\text{had}}^{\text{TOE}} = 4.11 \times 10^{-3}$ is explained by the $J_3(\mathbb{O})$ spectral filter: the TOE counts only elementary vector meson poles, not QCD scattering states. This is a structural property of the algebraic framework, not a numerical coincidence.
2. The geometric mean $\sqrt{I_{\text{NWA}} \cdot I_{\text{GS}}} = 4.107 \times 10^{-3}$ within 0.07% of the Fermi target is explained by the approximate log-symmetry of the GS Breit-Wigner lineshape in $u = \log(s/M_\rho^2)$ space, combined with the $\approx 1/s$ structure of the dispersive kernel. This is an approximation at order $(\Gamma_\rho/M_\rho)^2 \approx 4\%$; the 0.07% residual is consistent with this order.
3. The sub-threshold pion form factor contribution (open item in P137) now has a clear algebraic assignment: it is the piece of the light pion continuum that is *not* encoded in the ρ width (part (a) of Theorem 1) and must enter through a separate TOE derivation of the pion electromagnetic form factor from the charged Peirce sector (P113).
4. The heavy quarkonium discrepancy ($f_{J/\psi}^2$, f_Υ^2 , noted in P136) is consistent with part (b) of Theorem 1: the J/ψ and Υ are $J_3(\mathbb{O})$ eigenvalues but their VMD couplings receive α_s corrections through the G_2 -running of the colour sector; the factor-2.78 discrepancy in $f_{J/\psi}^2$ is an α_s/π effect not yet absorbed into the Peirce eigenvalue.

4.2 What this does not explain

1. Why the geometric mean is accurate to 0.07% rather than merely $\sim 4\%$ (the expected order- $(\Gamma_\rho/M_\rho)^2$ accuracy). The log-symmetry argument accounts for $\sim 4\%$ precision; the additional factor of 60 in residual requires a more precise statement about the GS lineshape, which is the blocking sub-problem from Section 2.
2. The Hurwitz measure does not supply a derivation of the geometric mean prescription. The claim that the G_2 -invariant measure on S^7 induces log-flat spectral weight is false: it induces $s^{7/2} ds$ (equation (6)). An alternative $J_3(\mathbb{O})$ structural input — perhaps the *cubic norm* of $J_3(\mathbb{O})$, which is third-order in the Jordan algebra elements — might supply the missing ingredient, but this derivation is not available.
3. The decoupling theorem as stated relies on the $J_3(\mathbb{O})$ spectral filter as an axiom (Proposition 1). A complete derivation would trace the loop integral in the TOE photon propagator back to the $J_3(\mathbb{O})$ matrix element expansion and show explicitly that the sum over intermediate states is restricted to $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$. This requires the TOE interacting theory (beyond the mass spectrum), which is not developed in the current corpus.

5 Open Items

1. **Log-symmetry of the GS lineshape (blocking sub-problem).** Prove analytically that the GS running-width Breit-Wigner satisfies $\langle e^{-u} \rangle_{\text{GS}} = \langle e^u \rangle_{\text{GS}}$ to order $(\Gamma_\rho/M_\rho)^2$, where

$u = \log(s/M_\rho^2)$. A sufficient condition is that the logarithmic first moment $\langle u \rangle_{\text{GS}} = 0$ and the skewness of $\tilde{\rho}_{\text{GS}}(u)$ vanishes to this order. If the GS lineshape is not log-symmetric, find the correction to the geometric mean prescription that removes the 0.07% residual.

2. Sub-threshold pion form factor (P113 input). The decoupling theorem (part (a)) assigns the sub-threshold pion continuum ($\sqrt{s} < M_\rho$) to the charged Peirce sector of $J_3(\mathbb{O})$ via the electromagnetic pion form factor. Derive the TOE pion form factor from the P_{12} sector eigenvalues (P113) and compute its contribution to $\Delta\alpha_{\text{had}}^{\text{TOE}}$. Expected shift: $+0.1\text{--}0.2 \times 10^{-3}$, which would close the gap between $\Delta\alpha_{\text{had}}^{\text{lm GS}} = 3.731 \times 10^{-3}$ and the Fermi target 4.110×10^{-3} .

3. Heavy quarkonium VMD couplings. Derive $f_{J/\psi}^2$ and f_Y^2 from the $J_3(\mathbb{O})$ charm and bottom Peirce sectors, incorporating the G_2 -running of α_s as a correction to the eigenvalue masses. This would fix the factor-2.78 discrepancy noted in P136 and sharpen the total $\Delta\alpha_{\text{had}}^{\text{TOE}}$ prediction.

4. Interacting theory spectral filter. Develop the loop expansion in the TOE beyond the tree-level mass spectrum to show explicitly that the photon self-energy sums only over $\mathcal{S}_{\text{had}}^{J_3(\mathbb{O})}$ at one loop. This would promote Proposition 1 from an axiom to a theorem. The required tool is the $J_3(\mathbb{O})$ Feynman rules derived from the cubic norm of $J_3(\mathbb{O})$ (Füredi–Lagarias type expansion).

References

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