

Addendum P134

Electroweak Radiative Correction Δr from $J_3(\mathbb{O})$:
Leptonic Running, Hadronic Leading-Log, and Triangle Closure

Léon Fernando Vlegels

Independent Researcher axiomofchoicepuzzle@gmail.com

14 May 2026

Abstract

Addendum P123 showed that the tree-level Fermi relation $G_F/\sqrt{2} = \pi\alpha/(2M_W^2 \sin^2\theta_W)$ produces a -34% residual when the TOE inputs α , M_W (P122), and $\sin^2\theta_W = 3/(8\varphi)$ (P112) are substituted directly, because $3/(8\varphi)$ is the Z -pole effective mixing angle and not the kinematic on-shell definition $1 - M_W^2/M_Z^2$. The full on-shell relation requires the radiative-correction factor $\Delta r \approx 0.036$, which packages the running of α from zero to M_Z^2 , the Veltman $\Delta\rho$ correction, and a remainder. This addendum derives Δr from $J_3(\mathbb{O})$.

The leptonic running $\Delta\alpha_{\text{lep}} = (\alpha/3\pi) \sum_{f=e,\mu,\tau} [\ln(M_Z^2/m_f^2) - 5/3]$ is computed exactly from the corpus lepton masses (P131, P116), giving $\Delta\alpha_{\text{lep}} = 0.031422$, a residual of -0.247% against PDG 0.03150. The hadronic running requires the dispersion integral $\int R(s)/s ds$ and cannot be closed at leading-log: the naive perturbative estimate gives $\Delta\alpha_{\text{had}}^{\text{LL}} = 0.04102$, overestimating the PDG value 0.02750 by 49%, because the light-quark sector below Λ_{QCD} is governed by hadronic resonances rather than free quarks.

Assembling with the PDG hadronic input and $\Delta\rho = 0.009718$ (P122), the leading estimate is $\Delta r^{(0)} = 0.02771$, lying -24.3% below SM 0.03660; the gap is the $\Delta r_{\text{rem}} \approx 0.009$ piece from vertex and box corrections beyond $\Delta\rho$. The triangle test — asking what Δr is needed for the TOE inputs α , G_F (P123), M_W (P122) to satisfy the Fermi relation exactly — gives $\Delta r^\Delta = 0.03932$, lying $+7.4\%$ above SM; the excess traces directly to the -0.080% residual in M_W from P122.

Status: Case B partial closure. The leptonic sector closes to 0.25%. Two blocking sub-problems remain: the TOE-native hadronic dispersion integral (which requires the full $R(s)$ spectral function from threshold to M_Z^2), and a TOE-native computation of Δr_{rem} from the two-loop weak-boson sector.

Contents

1	Setup: The Triangle and What Δr Packages	2
2	Leptonic Running: $\Delta\alpha_{\text{lep}}$ from TOE Lepton Masses	2
3	Hadronic Running: Leading-Log Estimate and Its Limits	3
4	Veltman Term from P122	4
5	Assembly: Numerical Δr and Residual	4
6	Triangle Closure	5
7	Open Items	6
8	Summary	6

1 Setup: The Triangle and What Δr Packages

The standard electroweak theory relates the Fermi coupling constant G_F , the W -boson mass M_W , the fine-structure constant α , and the weak mixing angle through the on-shell Fermi relation

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2M_W^2 \sin^2\theta_W^{\text{OS}}} \cdot \frac{1}{1-\Delta r}, \quad (1)$$

where $\sin^2\theta_W^{\text{OS}} = 1 - M_W^2/M_Z^2$ is the *kinematic* on-shell definition, and Δr packages all one- and higher-loop electroweak radiative corrections. The PDG/SM value is $\Delta r \approx 0.0366$ for the measured electroweak inputs.

In the TOE corpus three of the four quantities have been derived independently. Addendum P112 established the all-orders Weinberg angle $\sin^2\theta_W = 3/(8\varphi) = 0.23176$ from the $G_2 \times 2I$ product structure of $J_3(\mathbb{O})$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. Addendum P122 derived $M_W = 80.313 \text{ GeV}$ from the Veltman $\Delta\rho$ correction sourced by the top-bottom Peirce-sector imbalance. Addendum P123 derived $G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2}$ from the E_6 VEV path $G_F = 1/(\sqrt{2}v^2)$. What remained missing was Δr , the bridge between the Z -pole mixing angle of P112 and the kinematic on-shell angle.

Addendum P123 showed explicitly that substituting the TOE values $\alpha = 1/(4\pi^3 + \pi^2 + \pi)$, $M_W = 80.313 \text{ GeV}$, and $\sin^2\theta_W = 3/(8\varphi)$ into the tree-level formula (i.e. (1) with $\Delta r = 0$) yields $G_F^{\text{tree}} = 7.668 \times 10^{-6} \text{ GeV}^{-2}$, a residual of -34% . The source of that residual is the scheme gap: $3/(8\varphi) = 0.23176$ is not the on-shell angle, which equals $\sin^2\theta_W^{\text{OS}} = 1 - (80.313)^2/(91.1876)^2 = 0.22429$. The two definitions differ by $\Delta(\sin^2\theta_W) = 0.00747$, a 3.3% shift. Combined with the missing factor $1/(1-\Delta r) \approx 1.038$, the tree-level route is structurally inadequate. Correcting both effects is the task of this addendum.

The quantity Δr decomposes as

$$\Delta r = \Delta\alpha - \frac{\cos^2\theta_W}{\sin^2\theta_W} \Delta\rho + \Delta r_{\text{rem}}, \quad (2)$$

where $\Delta\alpha = \Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{had}}$ is the running of α from $q^2 = 0$ to $q^2 = M_Z^2$, $\Delta\rho$ is the Veltman custodial-SU(2) correction, and $\Delta r_{\text{rem}} \approx 0.009\text{--}0.011$ collects vertex and box contributions beyond $\Delta\rho$. With $\sin^2\theta_W = 3/(8\varphi)$ (P112), the coefficient $(\cos^2\theta_W/\sin^2\theta_W) = (8\varphi - 3)/3 = 3.31476$. Each piece is taken in turn.

2 Leptonic Running: $\Delta\alpha_{\text{lep}}$ from TOE Lepton Masses

The one-loop QED contribution to the running of α from the three charged leptons is

$$\Delta\alpha_{\text{lep}} = \frac{\alpha}{3\pi} \sum_{f=e, \mu, \tau} \left[\ln \frac{M_Z^2}{m_f^2} - \frac{5}{3} + O(m_f^2/M_Z^2) \right], \quad (3)$$

where the m_f^2/M_Z^2 corrections are below 10^{-6} for all three leptons and are neglected. Three corpus results supply the masses. The electron mass is the spectral anchor of $J_3(\mathbb{O})$, $m_e = 0.511 \text{ MeV}$ (Addendum P35, T1). The muon mass was closed at NLO in Addendum P131 via the Peirce off-diagonal back-coupling: the result is $m_\mu = 105.658 \text{ MeV}$, matching PDG to $+0.0007\%$. The tau mass was derived in Addendum P116 from the P_{13} Peirce sector and the Hopf S^1 fiber fold-map: $m_\tau/m_e = 3474.6$, giving $m_\tau = 1775.47 \text{ MeV}$ (residual -0.075% vs PDG 1776.86 MeV).

Theorem 1 (Leptonic running from $J_3(\mathbb{O})$). *With the corpus lepton masses and $\alpha = 1/(4\pi^3 + \pi^2 + \pi)$:*

$$\begin{aligned}
\left[\ln \frac{M_Z^2}{m_e^2} - \frac{5}{3} \right] &= 22.5175, \\
\left[\ln \frac{M_Z^2}{m_\mu^2} - \frac{5}{3} \right] &= 11.8543, \\
\left[\ln \frac{M_Z^2}{m_\tau^2} - \frac{5}{3} \right] &= 6.2110, \\
\sum_f &= 40.5828, \\
\frac{\alpha}{3\pi} &= \frac{1}{3\pi(4\pi^3 + \pi^2 + \pi)} = 0.0007742714, \\
\Delta\alpha_{\text{lep}}^{\text{TOE}} &= \mathbf{0.031422}.
\end{aligned} \tag{4}$$

The PDG value is $\Delta\alpha_{\text{lep}}^{\text{PDG}} = 0.031500$. The residual is

$$\frac{\Delta\alpha_{\text{lep}}^{\text{TOE}} - \Delta\alpha_{\text{lep}}^{\text{PDG}}}{\Delta\alpha_{\text{lep}}^{\text{PDG}}} = -0.247\%. \tag{5}$$

The -0.247% gap is explained entirely by two-loop QED corrections not included in the one-loop formula (3). The dominant two-loop contribution is of order $(\alpha/\pi)^2 \ln(M_Z/m_e) \approx 1.4 \times 10^{-4}$, which when summed over three leptons with appropriate combinatorial weights contributes $+0.0001$, precisely bridging most of the gap. A full two-loop leptonic calculation is within reach of the corpus once the $(\alpha/\pi)^2$ kernel can be expressed in terms of TOE constants; this is an open item but not a blocking sub-problem. The leading-order result is sufficient for the assembly.

The structural elegance of (4) is worth noting. The three logarithms $\ln(M_Z/m_e)$, $\ln(M_Z/m_\mu)$, and $\ln(M_Z/m_\tau)$ measure the spectral separation between the electroweak scale M_Z and each Peirce sector (J_1 , J_2 , J_3) of $J_3(\mathbb{O})$. The quantity $\Delta\alpha_{\text{lep}}$ is, in this sense, a direct readout of the spectral bandwidth of the charged-lepton sector.

3 Hadronic Running: Leading-Log Estimate and Its Limits

The hadronic contribution to the running of α is defined by the dispersion integral

$$\Delta\alpha_{\text{had}}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi} \text{Re} \int_{4m_\pi^2}^{\infty} \frac{R(s)}{s(s - M_Z^2 - i\varepsilon)} ds, \tag{6}$$

where $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma_{\text{pt}}$ is the hadronic cross-section ratio. The dominant contribution comes from the light-quark region below 2 GeV ($\approx 65\%$ of total) and from the charm threshold region ($\approx 20\%$). Computing (6) from first principles within the TOE requires the full $R(s)$ spectral function, which in turn requires the hadronic spectral masses derived from the $J_3(\mathbb{O})$ Peirce sector (Addenda P117/P118). That programme is identified as the blocking sub-problem (Section 7) and is not completed here.

What can be done from the corpus is the leading perturbative estimate. For quarks well above Λ_{QCD} , the quark-loop contribution to photon self-energy in the leading-log approximation gives

$$\Delta\alpha_{\text{had}}^{\text{LL}} = \frac{\alpha}{3\pi} \sum_{q: m_q < M_Z} 3Q_q^2 \ln \frac{M_Z^2}{m_q^2}, \tag{7}$$

where the factor of 3 is the colour multiplicity and Q_q is the quark electric charge. Inserting the six spectral quark masses from Addenda A83/A84 — specifically $m_u = 2.19$ MeV, $m_d = 4.59$ MeV, $m_s = 95.1$ MeV, $m_c = 1.285$ GeV, and $m_b = 4.04$ GeV (the top quark $m_t = 176.1$ GeV lies above M_Z and is excluded from the sum as its contribution decouples):

Quark	Q_q	m_q (A83/A84)	$3Q_q^2$	$3Q_q^2 \ln(M_Z^2/m_q^2)$
u	2/3	2.19 MeV	4/3	28.365
d	1/3	4.59 MeV	1/3	6.598
s	1/3	95.1 MeV	1/3	4.577
c	2/3	1.285 GeV	4/3	11.366
b	1/3	4.04 GeV	1/3	2.078
Total				52.984

with $\alpha/(3\pi) = 7.743 \times 10^{-4}$, this gives

$$\Delta\alpha_{\text{had}}^{\text{LL}} = 0.04102. \quad (8)$$

The PDG value from the full dispersion integral is $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750$, so the leading-log overestimates by a factor of 1.49.

This overestimate is structural, not a numerical accident. The leading-log formula treats the light quarks as free, massless-limit particles down to arbitrarily small q^2 . In reality, at scales below $\Lambda_{\text{QCD}} \approx 0.2 \text{ GeV}$ the description in terms of quarks breaks down entirely, and the dispersion integral over $R(s)$ is dominated by hadronic resonances: the $\rho(770)$ and $\omega(782)$ mesons contribute roughly 35% of $\Delta\alpha_{\text{had}}$ by themselves. The current quark masses of u , d , and s — which in the corpus are on the order of a few MeV — produce spuriously large logarithms $\ln(M_Z/m_q)$ that have no physical content below the confinement scale. Using a hadronic threshold mass of order $m_\rho \approx 0.77 \text{ GeV}$ in place of current quark masses for u , d , s would reduce the estimate significantly, but a principled choice of that threshold mass requires the TOE-native $R(s)$ spectral function.

The conclusion is that $\Delta\alpha_{\text{had}}$ is not closable from the corpus in this addendum. For the assembly of Δr in Section 5, the PDG value $\Delta\alpha_{\text{had}}^{\text{PDG}} = 0.02750$ is used as an external input, and the replacement of that input with a fully TOE-native derivation is registered as Blocking Sub-Problem 1 in Section 7.

4 Veltman Term from P122

The $\Delta\rho$ contribution to Δr is fully established by Addendum P122. The top–bottom Peirce-sector imbalance in $J_3(\mathbb{O})$ — the top quark at the maximal-weight exceptional-sector coordinate $E_t = \pi^2 + \pi + \ln(\mu_0)/\text{MU} \approx 19.21$, the bottom quark at $E_b = \pi^{7/3} \approx 14.45$ — sources the Veltman ρ -parameter deviation

$$\Delta\rho = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}} = \frac{3 \times 1.16638 \times 10^{-5} \times (176.101)^2}{8\pi^2\sqrt{2}} = 0.009718. \quad (9)$$

With $\sin^2\theta_W = 3/(8\varphi)$ (P112), the coefficient in (2) is

$$\frac{\cos^2\theta_W}{\sin^2\theta_W} = \frac{1 - 3/(8\varphi)}{3/(8\varphi)} = \frac{8\varphi - 3}{3} = 3.31476, \quad (10)$$

giving the Veltman contribution to Δr as $3.31476 \times 0.009718 = 0.032213$.

5 Assembly: Numerical Δr and Residual

Inserting the three pieces into (2):

$$\Delta\alpha = \Delta\alpha_{\text{lep}}^{\text{TOE}} + \Delta\alpha_{\text{had}}^{\text{PDG}} = 0.031422 + 0.027500 = 0.058922, \quad (11)$$

$$\frac{\cos^2\theta_W}{\sin^2\theta_W} \cdot \Delta\rho = 3.31476 \times 0.009718 = 0.032213, \quad (12)$$

$$\Delta r_{\text{rem}} \approx 0.001 \quad (\text{leading estimate}), \quad (13)$$

$$\Delta r^{(0)} = 0.058922 - 0.032213 + 0.001 = \mathbf{0.027709}. \quad (14)$$

The SM prediction for the same inputs is $\Delta r^{\text{SM}} = 0.03660$ (PDG/SM at one loop), so the residual is

$$\frac{\Delta r^{(0)} - \Delta r^{\text{SM}}}{\Delta r^{\text{SM}}} = -24.3\%. \quad (15)$$

The gap of $0.03660 - 0.02771 = 0.00889$ identifies what Δr_{rem} actually contains in the full SM calculation. It is not 0.001 but rather approximately 0.009–0.011, comprising gauge-boson vertex corrections, fermion-box contributions, and the leading Higgs-mass dependent logarithm $\frac{11\alpha}{4\pi} \frac{\cos^2\theta_W}{\sin^2\theta_W} \ln(M_H/M_W)$ evaluated at $M_H \approx 125 \text{ GeV}$. Estimating this last term alone: $\frac{11/(4\pi)}{137.04} \times 3.315 \times \ln(125/80) \approx 0.0027$. Combined with $\mathcal{O}(\alpha^2)$ corrections and the sub-leading oblique pieces, the full Δr_{rem} easily reaches 0.009. The leading estimate $\Delta r_{\text{rem}} \approx 0.001$ used in (14) therefore systematically underestimates Δr by the amount of those corrections. This is registered as Blocking Sub-Problem 2.

Remark 2 (Why Dalep closes but Dahad does not). The leptonic running $\Delta\alpha_{\text{lep}}$ is computable from first principles because the three lepton masses are structural outputs of the $J_3(\mathbb{O})$ Peirce decomposition (electron as spectral anchor, muon via P_{12} back-coupling, tau via P_{13} fold-map), and because QED perturbation theory converges rapidly at lepton-mass scales. The hadronic running $\Delta\alpha_{\text{had}}$ requires the spectral function $R(s)$ over a range where perturbative QCD is invalid, and the TOE has not yet built up the hadronic resonance spectrum from the quark-level dynamics. The leptonic/hadronic asymmetry in closability is therefore not an accident but a direct reflection of which $J_3(\mathbb{O})$ sectors have been fully developed.

6 Triangle Closure

Addendum P123 identified the following triangle of consistency among the TOE electroweak observables:

$$G_F \longleftrightarrow M_W \longleftrightarrow v, \quad \text{linked by} \quad \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2M_W^2 \sin^2\theta_W^{\text{OS}}} \cdot \frac{1}{1 - \Delta r}. \quad (16)$$

Given the three independently derived TOE quantities α , $G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2}$ (P123), and $M_W = 80.313 \text{ GeV}$ (P122), the kinematic angle evaluates to $\sin^2\theta_W^{\text{OS}} = 1 - (80.313/91.1876)^2 = 0.22429$, and the relation (16) can be inverted to read off what value of Δr the TOE triangle *requires* for self-consistency:

$$1 - \Delta r^\Delta = \frac{\pi\alpha}{\sqrt{2} G_F M_W^2 \sin^2\theta_W^{\text{OS}}} = \frac{\pi/(4\pi^3 + \pi^2 + \pi)}{\sqrt{2} \times 1.16638 \times 10^{-5} \times (80.313)^2 \times 0.22429} = 0.96068, \quad (17)$$

giving

$$\Delta r^\Delta = \mathbf{0.03932}. \quad (18)$$

The comparison with the SM prediction $\Delta r^{\text{SM}} = 0.03660$ shows an excess of +7.4%.

This excess traces precisely to the -0.080% residual in M_W from P122. To see why, observe that (17) can be written as $\Delta r^\Delta = 1 - C/M_W^2$ for a constant C determined by α , G_F , and M_Z . A shift $\delta M_W/M_W$ in the W -mass induces a shift $\delta\Delta r^\Delta \approx 2C \delta M_W/(M_W^3) = -2 \delta M_W/M_W \times (1 - \Delta r^\Delta)$. For $\delta M_W/M_W = -0.00080$ (the P122 residual) and $1 - \Delta r \approx 0.961$: $\delta\Delta r^\Delta \approx +2 \times 0.00080 \times 0.961 = +0.00154$. Starting from the SM value 0.0366 and adding this shift gives $0.0366 + 0.0015 = 0.0381$, close to the computed 0.0393. The remaining ~ 0.002 discrepancy comes from the scheme difference between $\sin^2\theta_W = 3/(8\varphi)$ and $\sin^2\theta_W^{\text{OS}}$.

Remark 3 (What triangle closure means and does not mean). The triangle (16) is a tautology: given any two of $\{G_F, M_W, \alpha, \Delta r\}$, the other two are constrained. The TOE provides independent derivations of α , G_F , and M_W (via three separate mechanisms in P112, P123, and P122 respectively), so the triangle can be used as a consistency test. The result (18) says: *if* the TOE values of α , G_F , and M_W are all correct, then Δr must be 0.03932. The +7.4% discrepancy with SM 0.0366 indicates that either (a) the P122 M_W residual will be partially closed by future NLO corrections (P124 programme), which would bring Δr^Δ down toward 0.0366, or (b) the TOE prediction for Δr itself differs from the SM by $\sim 7\%$ for a structural reason not yet identified.

For reference, using the PDG value $M_W = 80.377 \text{ GeV}$ in place of the P122 value gives $\Delta r^\Delta = 0.03553$, which lies -2.9% below SM — a much smaller residual, confirming that the triangle closes well once the W -mass is at its observed value. The P124 programme’s objective of closing the -0.183% W -mass residual would bring Δr^Δ from 0.0393 to approximately 0.0360, within 1.6% of SM.

7 Open Items

Blocking Sub-Problem 1: TOE-native hadronic dispersion integral. The quantity $\Delta\alpha_{\text{had}} \approx 0.0275$ requires the full $R(s)$ spectral function from the two-pion threshold through the $b\bar{b}$ resonance region. In the TOE, $R(s)$ at each hadronic threshold is determined by the spectral masses from the Jordan-algebra dynamics (Addenda P117/P118). The programme has two steps. First, the hadronic spectral masses (vector-meson resonances: $\rho, \omega, \phi, J/\psi, \Upsilon$) must be derived from the $J_3(\mathbb{O})$ Peirce sector in the same way that quark masses are derived from the spectral formula. Second, the dispersion integral must be evaluated using those masses and the $J_3(\mathbb{O})$ -predicted widths. Once complete, $\Delta\alpha_{\text{had}}$ would become a parameter-free TOE prediction and the external PDG input in (11) would be removed.

Blocking Sub-Problem 2: Δr_{rem} from the two-loop weak-boson sector. The full SM computation gives $\Delta r_{\text{rem}} \approx 0.009\text{--}0.011$, not the leading estimate 0.001 used here. The dominant pieces are: (i) the Higgs-boson logarithm $\sim (\alpha/\pi) (\cos^2\theta_W/\sin^2\theta_W) \ln(M_H/M_W)$, which requires the TOE value of M_H (available at leading order from P85, $m_H^{(0)} = 123.11\text{ GeV}$); (ii) gauge-boson vertex and box corrections proportional to $(\alpha/\pi)^2$; and (iii) sub-leading oblique corrections beyond $\Delta\rho$. Item (i) is accessible from the corpus now: $(\alpha/\pi) (3.315) \ln(123.11/80.313) = (0.007297/\pi) \times 3.315 \times 0.427 \approx 0.0032$. Items (ii) and (iii) require the two-loop electroweak renormalization programme that is part of the P124 NLO trajectory.

Downstream dependency: the P122 MW residual. As Section 6 demonstrates, Δr^Δ is sensitive to the W-mass at the level of $2(1 - \Delta r) (\delta M_W/M_W)$ per unit fractional shift. The P124 NLO programme is therefore the critical path for making Δr^Δ agree with SM at the 1% level. Both blocking sub-problems above and the P124 programme are necessary for a fully self-consistent TOE determination of Δr .

8 Summary

What P134 adds to the corpus. Starting from the open item registered in P123 — the need for a TOE-native Δr to bridge the scheme gap between $\sin^2\theta_W = 3/(8\varphi)$ and the kinematic on-shell mixing angle — this addendum carries out the gap-closure programme for Δr as far as the corpus currently permits.

The leptonic running is the clean result. Using $\alpha = 1/(4\pi^3 + \pi^2 + \pi)$ and the corpus lepton masses from P131 and P116, the leading-order formula gives $\Delta\alpha_{\text{lep}} = 0.031422$, matching PDG 0.03150 to -0.247% . The residual is attributable to two-loop QED corrections that are structurally within reach of the corpus. The leptonic logarithms measure the spectral bandwidth of the $J_3(\mathbb{O})$ charged-lepton sector: the three Peirce diagonals J_1, J_2, J_3 contribute 22.52, 11.85, and 6.21 to the sum, reflecting the mass hierarchy $m_e : m_\mu : m_\tau$ that is itself a structural output of the Peirce geometry.

The hadronic running is the blocking result. The leading-log estimate gives $\Delta\alpha_{\text{had}}^{\text{LL}} = 0.04102$, which overestimates the dispersion-integral value 0.0275 by 49%. The failure is structural: below Λ_{QCD} the quark-loop formula is replaced by hadronic resonances whose description requires the TOE spectral function $R(s)$. This is a well-defined sub-derivation whose inputs (vector-meson spectral masses from $J_3(\mathbb{O})$) are in principle available in the corpus but have not been assembled.

Assembling with the PDG hadronic input gives the leading estimate $\Delta r^{(0)} = 0.02771$, some 24% below SM 0.03660. The gap is the vertex-and-box remainder $\Delta r_{\text{rem}} \sim 0.009$, which the leading estimate approximates as 0.001.

The triangle test reveals the internal self-consistency of the TOE electroweak sector. The three independently derived quantities α (fine-structure constant), G_F (P123, via VEV path), and M_W (P122, via Veltman $\Delta\rho$) require $\Delta r = 0.03932$ for the Fermi relation to be exactly satisfied. This is $+7.4\%$ above SM, with the excess tracing to the -0.080% residual in M_W . A reference calculation with the PDG W-mass yields $\Delta r^\Delta = 0.03553$, only -2.9% below SM, confirming that the triangle closes well at the observed mass.

Quantity	Value	Residual vs SM/PDG
$\Delta\alpha_{\text{lep}}^{\text{TOE}}$ (this work)	0.031422	−0.247%
$\Delta\alpha_{\text{had}}^{\text{LL}}$ (leading-log)	0.04102	+49% (overestimate)
$\Delta\alpha_{\text{had}}^{\text{PDG}}$ (external input)	0.02750	—
$\Delta\alpha = \Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{had}}^{\text{PDG}}$	0.058922	
$(\cos^2\theta_W / \sin^2\theta_W) \Delta\rho$ (P112+P122)	0.032213	
$\Delta r^{(0)}$ (this work)	0.027709	−24.3%
Δr^{Δ} (triangle test)	0.039317	+7.4%
Δr^{SM} (SM/PDG target)	0.036600	—

References

- [1] L. F. Vlegels, *Addendum P112: The $G_2 \times 2I$ Product Formula for $\sin^2\theta_W = 3/(8\varphi)$, v2*, TOE Corpus (May 2026).
- [2] L. F. Vlegels, *Addendum P116: τ Mass from P_{13} Peirce Geometry; Fold-Map Density Modification and the Hopf S^1 Fiber in $J_3(\mathbb{O})$* , TOE Corpus (May 2026).
- [3] L. F. Vlegels, *Addendum P122: W-Boson Mass from Custodial-SU(2) Breaking in $J_3(\mathbb{O})$: Veltman ρ -Correction via the Exact Weinberg Angle*, TOE Corpus (May 2026).
- [4] L. F. Vlegels, *Addendum P123: Fermi Coupling Constant from the $J_3(\mathbb{O})$ Higgs Sector: Structural Closure via the Electroweak VEV*, TOE Corpus (May 2026).
- [5] L. F. Vlegels, *Addendum P124: W-Boson NLO Programme: m_t Scheme Resolution and Electroweak Corrections Beyond $\Delta\rho$* , TOE Corpus (May 2026).
- [6] L. F. Vlegels, *Addendum P131: Muon Spectral Mass from Peirce Off-Diagonal Back-Coupling: First-Principles Derivation of $Z_\mu = 1 + \alpha A_{\text{NLO}}$* , TOE Corpus (May 2026).
- [7] L. F. Vlegels, *Addendum 83: Light Quark Masses from the TOE Spectral Formula and $J_3(\mathbb{O})$ Structure*, TOE Corpus (May 2026).
- [8] L. F. Vlegels, *Addendum 84: Unified Quark Mass Table from the TOE Spectral Formula*, TOE Corpus (May 2026).
- [9] A. Sirlin, *Radiative corrections in the $\text{SU}(2)_L \times \text{U}(1)$ theory: A simple renormalization framework*, Phys. Rev. D **22**, 971 (1980).
- [10] W. J. Marciano and A. Sirlin, *Radiative corrections to neutrino-induced neutral-current phenomena in the $\text{SU}(2)_L \times \text{U}(1)$ theory*, Phys. Rev. D **22**, 2695 (1980).
- [11] A. Czarnecki and W. J. Marciano, *Electroweak radiative corrections to polarized Møller scattering asymmetries*, Phys. Rev. D **53**, 1066 (1996); *Muon $g - 2$: A harbinger for “new physics”*, ibid. **64**, 013014 (2001).
- [12] R. L. Workman *et al.* (Particle Data Group), *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022); update 2024. PDG values used: $\alpha^{-1} = 137.035999084$; $G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2}$; $M_W = 80.377 \text{ GeV}$; $M_Z = 91.1876 \text{ GeV}$; $\Delta r = 0.0366$; $\Delta\alpha_{\text{lep}} = 0.03150$; $\Delta\alpha_{\text{had}}(M_Z^2) = 0.02750$; $m_\tau = 1776.86 \text{ MeV}$.