

Addendum P133

Muon Mass NNLO: Two-Loop G_2 -Orbit Structure
and the $\sin(\pi/16)$ Structural Candidate

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Abstract

Addendum P131 closed the muon mass to NLO, leaving a residual of $+0.0007\%$ against the PDG value. It identified two NNLO topologies — each of order $(\alpha A_{\text{NLO}})^2 \approx 3.6 \times 10^{-5}$ — whose near-cancellation leaves the 7.0×10^{-6} target, but classified the exact computation as Case B pending the two-loop G_2 -orbit integral. This addendum runs the gap-closure loop on that sub-problem to its natural stopping point. We factor the near-cancellation as $(\alpha A_{\text{NLO}})^2 \cdot f$, determine the target fraction $f = 0.1943$, and examine every structurally motivated candidate. The best match is $f = \sin(\pi L_{G_2}^2) = \sin(\pi/16) = 0.19509$, arising from the squared G_2 loop eigenvalue $L_{G_2} = -\frac{1}{4}$ (P81); it gives $(\alpha A_{\text{NLO}})^2 \cdot \sin(\pi/16) = 7.03 \times 10^{-6}$, within 0.4% of the target and within the five-significant-figure precision floor of the corpus. The second candidate is the orbit-averaging suppression $1/\mathcal{N}_{\text{Fano}} = 1/5 = 0.200$, which gives 7.20×10^{-6} , 2.9% over the target but still within the precision floor. Neither candidate is derived from a first-principles two-loop integral; the blocking sub-problem is the evaluation of the G_2 -invariant two-loop integral over $S^6 = \text{Im}(\mathbb{O})/|\cdot|$. **Classification: Case B** — structural candidate $\sin(\pi/16)$ identified; blocking integral named. The muon mass open problem remains closed at NLO; the NNLO computation is not required for closure at current precision.

1 Setup: the P131 NNLO blocking sub-problem

The NLO muon mass derivation of P131 yielded

$$m_\mu^{\text{NLO}} = \frac{m_\mu^{\text{LO}}}{1 + \alpha A_{\text{NLO}}} = 105.659 \text{ MeV}, \quad \text{residual} = +0.0007\%, \quad (1)$$

against the PDG value $m_\mu = 105.6584 \text{ MeV}$. The fractional discrepancy $\Delta \equiv (m_\mu^{\text{NLO}} - m_\mu^{\text{PDG}})/m_\mu^{\text{PDG}} = 7.0 \times 10^{-6}$ represents a deficit of 7.0×10^{-6} in the wave-function renormalisation factor: Z_μ must increase by this amount to close the gap. P131 §5 argued that this residual lies within the five-significant-figure precision floor of the corpus inputs, and classified the NNLO computation as Case B. The present addendum runs the gap-closure loop on that computation as far as the existing corpus tools permit.

The NNLO Dyson structure is

$$Z_\mu^{\text{NNLO}} = 1 + \alpha A_{\text{NLO}} + \alpha^2 C_{\text{NNLO}}, \quad (2)$$

where C_{NNLO} is the net two-loop amplitude. For Z_μ^{NNLO} to close the gap, we need

$$\alpha^2 C_{\text{NNLO}} = \Delta = 7.0 \times 10^{-6}. \quad (3)$$

P131 Proposition 5.1 identified two topologies that contribute to C_{NNLO} . Topology A is the sequential double fold: the muon in sector $\mathcal{E}_{22}$ emits a $\mathcal{P}_{12}$ fluctuation into the electron sector,

which returns and emits a second fluctuation before the final return. This carries amplitude $C_A = A_{\text{NLO}}^2$, giving $\alpha^2 C_A = (\alpha A_{\text{NLO}})^2$. Topology B is the indirect path $\mathcal{E}_{22} \rightarrow \mathcal{P}_{12} \rightarrow \mathcal{E}_{11} \rightarrow \mathcal{P}_{13} \rightarrow \mathcal{E}_{33} \rightarrow \mathcal{P}_{23} \rightarrow \mathcal{E}_{22}$ through the tau sector, which carries amplitude $C_B \sim A_{\text{NLO}} \cdot f_{\text{bulk}}$ where $f_{\text{bulk}} = 4\pi^3/\alpha^{-1} = 0.905$. Both topologies enter at order α^2 , with opposite signs, and their net is $C_{\text{NNLO}} = C_A - C_B$ (sequential fold increases Z ; tau-sector loop decreases it).

The gap-closure loop begins by computing the scale of the individual topologies and then determining what fraction must survive after cancellation.

2 Numerical anatomy of the near-cancellation

The NNLO scale is set by Topology A:

$$\alpha^2 C_A = (\alpha A_{\text{NLO}})^2 = (6.002 \times 10^{-3})^2 = 3.602 \times 10^{-5}. \quad (4)$$

The tau-sector topology (Topology B) is estimated from its leading amplitude:

$$\alpha^2 C_B^{\text{est}} \sim \alpha^2 A_{\text{NLO}} f_{\text{bulk}} = 5.327 \times 10^{-5} \times 0.822 \times 0.905 = 3.964 \times 10^{-5}. \quad (5)$$

Both topologies are of order $\alpha^2 \sim 5 \times 10^{-5}$, as P131 stated. What is not determined from the leading amplitudes alone is their *net after the G_2 -orbit integration modifies each one*: the estimates (4)–(5) are the tree-level inputs to the two-loop integral, not the output. The output — the net C_{NNLO} — is constrained by the gap to satisfy $\alpha^2 C_{\text{NNLO}} = 7.0 \times 10^{-6}$.

Factoring out the Topology A scale, we write

$$\alpha^2 C_{\text{NNLO}} = \alpha^2 A_{\text{NLO}}^2 \cdot f = 3.602 \times 10^{-5} \cdot f, \quad (6)$$

where $f = C_{\text{NNLO}}/A_{\text{NLO}}^2$ is the surviving fraction. Equation (3) then fixes

$$f = \frac{7.0 \times 10^{-6}}{3.602 \times 10^{-5}} = 0.1943. \quad (7)$$

The gap-closure loop reduces to a single question: which TOE-native expression evaluates to 0.1943?

3 Structural candidates for the surviving fraction

The search proceeds by structural motivation, not numerology: each candidate must arise from an element of the TOE framework that has not yet been deployed in this context, at an order of magnitude consistent with the gap.

Candidate A: orbit-averaging suppression $1/\mathcal{N}_{\text{Fano}}$. At NLO, the orbit-averaging over $\mathcal{N}_{\text{Fano}} = 5$ Peirce positions (P99, Theorem T2) produces the suppression factor $4\lambda^2/\mathcal{N}_{\text{Fano}} = 4\lambda^2/5$ in the fold amplitude. The orbit dimension $\mathcal{N}_{\text{Fano}} = 5$ is the number of Peirce position types in $J_3(\mathbb{O})$ that the F_4/G_2 Fano loop action can reach from the G_2 -stabilised weight-4 centre E_{22} . At NNLO, if the near-cancellation of the two topologies is also governed by the same orbit-averaging — with Topology A and Topology B contributing equally to the averaged orbit sum and their residual after the $\mathcal{N}_{\text{Fano}}$ average surviving — then the fraction is simply $1/\mathcal{N}_{\text{Fano}} = 1/5$. Numerically:

$$\alpha^2 A_{\text{NLO}}^2 / \mathcal{N}_{\text{Fano}} = 3.602 \times 10^{-5} / 5 = 7.20 \times 10^{-6}, \quad \text{error vs. target: } +2.9\%. \quad (8)$$

This candidate is structurally natural — the same denominator that governs the NLO formula appears at NNLO — but overshoots the target by an amount that, while inside the precision floor, is the largest discrepancy among the candidates examined.

Candidate B: squared G_2 eigenvalue $\sin(\pi L_{G_2}^2)$. The G_2 loop eigenvalue is $L_{G_2} = -1/4$ (P81, Theorem T1), derived from the Peirce- $\frac{1}{2}$ product rule applied once around the G_2 orbit over $S^6 = \text{Im}(\mathbb{O})/|\cdot|$. At NLO, a single traversal of the orbit scales the fold amplitude by L_{G_2} . At NNLO, the orbit is traversed twice in the net diagram (after the Topology A/B cancellation), and the squared eigenvalue $L_{G_2}^2 = (1/4)^2 = 1/16$ enters. In the S^6 orbit geometry, the NLO correction is encoded as an angle $\lambda = \sin(\pi/14)$ through the Wolfenstein phase structure (P75, P81). If the two-loop phase is encoded analogously — with $L_{G_2}^2$ replacing L_{G_2} in the angular argument — the surviving fraction is

$$f = \sin(\pi L_{G_2}^2) = \sin(\pi/16) = 0.19509. \quad (9)$$

The structural argument: $\pi L_{G_2}^2 = \pi \times 1/16 = \pi/16$ is the phase angle accumulated by the two-loop G_2 -orbit traversal; the sine of this angle is the projection of the two-loop amplitude onto the propagation direction in the S^3 sector. Note that the Bryant–Fano loop eigenvalue $L_{\text{Fano}} = +1/4$ (P98) satisfies $L_{\text{Fano}}^2 = L_{G_2}^2 = 1/16$, so both two-loop eigenvalues converge to the same squared value — consistent with the two-channel $O(\lambda^2)$ protection of P98 extending to the NNLO sector. Numerically:

$$\alpha^2 A_{\text{NLO}}^2 \cdot \sin(\pi/16) = 3.602 \times 10^{-5} \times 0.19509 = 7.03 \times 10^{-6}, \quad \text{error vs. target: } +0.4\%. \quad (10)$$

Label	Expression	Value	$\alpha^2 A_{\text{NLO}}^2 \cdot f$	Error
Target	—	0.1943	7.00×10^{-6}	—
Candidate A	$1/\mathcal{N}_{\text{Fano}} = 1/5$	0.2000	7.20×10^{-6}	+2.9%
Candidate B	$\sin(\pi L_{G_2}^2) = \sin(\pi/16)$	0.19509	7.03×10^{-6}	+0.4%
$1/(2\phi^2)$	Golden section	0.19098	6.88×10^{-6}	−1.7%
$\tan(\pi/16)$	—	0.19891	7.17×10^{-6}	+2.4%

Table 1: Structural candidates for the surviving fraction $f = C_{\text{NNLO}}/A_{\text{NLO}}^2$. All errors are within the precision floor ($\sim 10^{-5}$) of the corpus inputs, so no candidate can be distinguished numerically at current precision. Candidate B ($\sin(\pi/16)$ from $L_{G_2}^2 = 1/16$) has the smallest error and the strongest structural motivation.

Among all candidates examined (Table 1), Candidate B gives the best numerical match and the most direct derivation from existing corpus results. Its structural argument uses only $L_{G_2} = -1/4$ (P81, proved) and the S^6 -orbit phase encoding convention (which also underpins $\lambda = \sin(\pi/14)$ in P75); no new TOE element is introduced.

Proposition 3.1 (Structural conjecture: two-loop eigenvalue encoding). *If the net NNLO correction to Z_μ is $\alpha^2 A_{\text{NLO}}^2 \cdot f$, then the TOE-native expression for f is*

$$f_{\text{conj}} = \sin(\pi L_{G_2}^2) = \sin(\pi/16) = 0.19509, \quad (11)$$

where $L_{G_2} = -1/4$ is the G_2 loop eigenvalue of P81. The corresponding NNLO correction to Z_μ is

$$\alpha^2 A_{\text{NLO}}^2 \sin(\pi/16) = 7.03 \times 10^{-6}, \quad (12)$$

within 0.4% of the target 7.0×10^{-6} and within the five-significant-figure precision floor of the corpus. This is a conjecture, not a theorem: the derivation of $\sin(\pi L_{G_2}^2)$ from the two-loop G_2 -orbit integral has not been established.

Were Proposition 3.1 established, the NNLO mass would be

$$m_\mu^{\text{NNLO}} = \frac{m_\mu^{\text{LO}}}{1 + \alpha A_{\text{NLO}} + \alpha^2 A_{\text{NLO}}^2 \sin(\pi/16)} = 105.6583 \text{ MeV}, \quad (13)$$

with residual -1.3×10^{-6} against the PDG value — an overclosure by approximately one part in a million, well within the precision floor and consistent with the precision artefact interpretation of P131 §5.

4 The blocking sub-problem: the two-loop G_2 -orbit integral

The candidates of Section 3 are structurally motivated but not derived. Converting Proposition 3.1 from a conjecture to a theorem requires evaluating the two-loop G_2 -orbit integral, which we now state precisely.

At NLO, the G_2 -invariant integral over the G_2 orbit on S^6 with the round measure $d\mu_{S^6}$ of a phase function encoding the Peirce $\frac{1}{2}$ -coupling between $\mathcal{E}_{11}$ and $\mathcal{E}_{22}$ evaluates to A_{NLO} (up to the $4\lambda^2/5$ correction from the weight-4 centre geometry, P99). In schematic form:

$$\int_{G_2\text{-orbit}} \phi(x) d\mu_{S^6}(x) = A_{\text{NLO}} = A_{\text{LO}} \left(1 - \frac{4\lambda^2}{\mathcal{N}_{\text{Fano}}} \right), \quad (14)$$

where $\phi(x)$ is the Hopf-phase amplitude function encoding the $\mathcal{P}_{12}$ propagation from $\mathcal{E}_{22}$ to $\mathcal{E}_{11}$ and back. At NNLO, the two-loop diagram replaces ϕ with ϕ^2 (for Topology A, the sequential double fold) and with a crossed-phase function ψ (for Topology B, the tau-sector loop), and both integrals are performed simultaneously with the G_2 -equivariant measure:

$$C_{\text{NNLO}} = \int_{G_2\text{-orbit}} \phi(x)^2 d\mu_{S^6}(x) - \int_{G_2\text{-orbit}} \psi(x) d\mu_{S^6}(x). \quad (15)$$

The first integral (Topology A) evaluates the squared Hopf-phase amplitude over the G_2 orbit. The second (Topology B) evaluates the crossed amplitude ψ , which involves the additional $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$ propagators through the tau sector. Neither integral has been computed.

The G_2 -invariant structure of the first integral constrains its value: the G_2 orbit over S^6 has dimension 6, and the eigenvalue per traversal $L_{G_2} = -1/4$ (P81) determines the leading contribution as A_{NLO}^2 times a two-loop correction. The conjecture $f = \sin(\pi L_{G_2}^2)$ is equivalent to asserting that the ratio of the two-loop orbit integral to its tree-level estimate A_{NLO}^2 is $\sin(\pi/16)$, which is further equivalent to asserting that the two-loop G_2 -orbit integral over S^6 satisfies

$$\frac{C_{\text{NNLO}}}{A_{\text{NLO}}^2} = \sin(\pi L_{G_2}^2), \quad (16)$$

or alternatively $C_{\text{NNLO}} = A_{\text{NLO}}^2 \sin(\pi/16)$. This is the precise form of the blocking sub-problem.

The second candidate ($1/\mathcal{N}_{\text{Fano}} = 1/5$) corresponds to the assertion $C_{\text{NNLO}}/A_{\text{NLO}}^2 = 1/5$, which would follow if the G_2 -orbit integration of the two-loop amplitude distributes equally over the $\mathcal{N}_{\text{Fano}} = 5$ active Peirce positions and the net is the mean residual after cancellation. This is a natural generalisation of the NLO orbit-averaging argument of P99, but it is also not derived. The two candidates are numerically indistinguishable at current precision (Table 1) and require the exact evaluation of (16) to separate.

5 Classification and open items

Status of the gap. The residual of +0.0007% in the NLO muon mass is within the precision floor of the current corpus inputs, as P131 §5 argued. It does not represent a missing structural term; the muon mass is closed at NLO. The NNLO computation is not required for closure.

Classification of P133. Case B. The structural mechanism for the NNLO correction is identified: it arises from the near-cancellation of two G_2 -orbit topologies at order α^2 , each of magnitude $\sim 3.6 \times 10^{-5}$, with a net of $\sim 7 \times 10^{-6}$ encoding the surviving fraction f . The best structural candidate is $\sin(\pi/16)$ from the squared loop eigenvalue $L_{G_2}^2 = 1/16$; the orbit-averaging candidate $1/\mathcal{N}_{\text{Fano}}$ is a second structurally motivated choice. The blocking sub-problem is the evaluation of the two-loop G_2 -orbit integral (15) to sufficient precision to distinguish the candidates.

Open items.

OI-P133-1 (blocking integral). Evaluate the two-loop G_2 -orbit integral $\int_{G_2\text{-orbit}} \phi(x)^2 d\mu_{S^6}(x)$ and its tau-sector counterpart $\int_{G_2\text{-orbit}} \psi(x) d\mu_{S^6}(x)$ to establish whether $C_{\text{NNLO}}/A_{\text{NLO}}^2 = \sin(\pi/16)$ or $1/\mathcal{N}_{\text{Fano}}$ or neither. This requires the explicit form of the Hopf-phase amplitude ϕ and the crossed amplitude ψ in terms of the G_2 -invariant measure on S^6 .

OI-P133-2 (angular encoding at two loops). At one loop, the G_2 -orbit phase is encoded by the angle $\pi/14$ through $\lambda = \sin(\pi/14)$. At two loops, the proposed encoding is $\pi/16$ through $L_{G_2}^2 = 1/16$. A general derivation of the angular encoding rule for the n -loop G_2 -orbit integral would confirm or refute the $\sin(\pi L_{G_2}^2)$ conjecture for all loop orders, and would clarify whether $\pi/14$ and $\pi/16$ arise from the same underlying mechanism or from distinct structures.

OI-P133-3 (candidate discrimination). The two candidates differ by 2.5% in the predicted fraction f ($\sin(\pi/16) = 0.1951$ vs. $1/\mathcal{N}_{\text{Fano}} = 0.2000$), corresponding to a difference of 1.7×10^{-7} in the NNLO correction to Z . Discriminating them numerically requires the corpus inputs to be specified to six or more significant figures — beyond the current precision floor. Improving the input precision of m_e , $\mathcal{M}$, and A_{NLO} to six significant figures is a prerequisite for using the muon mass as an NNLO discriminator.

Addendum	Result	Status
P113	NLO closure numerically: $m_\mu^{\text{NLO}} = 105.659 \text{ MeV}$, $+0.0007\%$	Closed (A)
P131	First-principles derivation of $Z_\mu = 1 + \alpha A_{\text{NLO}}$	Closed (A)
P131 §5	NNLO: two topologies $\sim 3.6 \times 10^{-5}$, net unknown	Open (B)
P133	NNLO: $f = 0.1943$; best candidate $\sin(\pi/16)$ from $L_{G_2}^2 = 1/16$	Structural (B)
OI-P133-1	Two-loop G_2 -orbit integral $\int \phi^2 d\mu_{S^6}$	Open
OI-P133-2	Angular encoding rule at n loops	Open
OI-P133-3	Candidate discrimination ($\sin(\pi/16)$ vs. $1/5$)	Needs 6sf inputs

References

- [1] L. F. Vlegels, *Addendum 75: Wolfenstein λ from G_2* , TOE Corpus (2025). $\lambda = \sin(\pi/14)$; $A_{\text{LO}} = \cos^2(\pi/14) \cos(2\pi/14)$.
- [2] L. F. Vlegels, *Addendum 81: Loop Eigenvalue $L_{G_2} = -1/4$ from the Peirce- $\frac{1}{2}$ Product; NLO Wolfenstein A* , TOE Corpus (2025). $L_{G_2} = -1/4$; $A_{\text{NLO}} = A_{\text{LO}}(1 - 4\lambda^2/5)$.
- [3] L. F. Vlegels, *Addendum 98: Bryant-Fano NLO Correction and the Dual Loop Pair*, TOE Corpus (2026). $L_{\text{Fano}} = +1/4$; two-channel $O(\lambda^2)$ protection.
- [4] L. F. Vlegels, *Addendum 99: The Weight-4 Centre, $\mathcal{N}_{\text{Fano}} = 5$ from First Principles*, TOE Corpus (2026). E_{22} uniquely stabilised; $\mathcal{N}_{\text{Fano}} = 6 - 1 = 5$ from Peirce position count.

- [5] L. F. Vlegels, *Addendum 113: Muon Mass EM Renormalisation*; $m_\mu/m_e = \exp(\mathcal{M}(\pi^2 - \pi))/(1 + \alpha A_{\text{NLO}})$, TOE Corpus (2026). NLO closure; OI-1 (first-principles Z_μ) left open.
- [6] L. F. Vlegels, *Addendum 131: Muon Spectral Mass from Peirce Off-Diagonal Back-Coupling*, TOE Corpus (2026). First-principles $Z_\mu = 1 + \alpha A_{\text{NLO}}$ from $\mathcal{P}_{12}(\mathbb{O})$; NNLO blocking subproblem Case B.
- [7] R. L. Workman *et al.* (Particle Data Group), *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022); updated 2024. $m_\mu = 105.6584 \text{ MeV}$.