

Structural Origin of the Matching Scale $\mu_{\text{match}} \approx 706 \text{ MeV}$

Addendum 132 to the TOE Corpus

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Abstract

Addendum 127 resolved the asymptotic obstruction in the strong-coupling series by introducing a matching scale μ_{match} where $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$, deriving

$$\mu_{\text{match}} = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right) \approx 705.9 \text{ MeV},$$

and leaving the structural origin of the exponent as an open item. This addendum closes that gap. The factor $(6 - \sqrt{3})$ in the numerator is a pure G_2 identity: $6 = |\Phi^+(G_2)|$ is the number of positive roots in G_2 , and $\sqrt{3}$ is the long-to-short root-length ratio ρ_{G_2} , so $(6 - \sqrt{3}) = \rho_{G_2}(2\rho_{G_2} - 1)$ with $2\rho_{G_2}^2 = |\Phi^+(G_2)|$. The denominator $23 = 3b_0$ is the $J_3(\mathbb{O})$ Casimir contribution to the $n_f = 5$ one-loop $\overline{\text{MS}}$ coefficient derived in Addendum 121. The prefactor 2π arises from the squared scale ratio in the one-loop running equation — equivalently, the double winding of the Hopf map that appears in the TOE lapse $u = \cos(2\beta)$. Together, the exponent reads

$$\frac{2\pi \rho_{G_2}(2\rho_{G_2} - 1)}{3b_0} = \frac{2\pi(6 - \sqrt{3})}{23},$$

a formula involving only G_2 root data and the $J_3(\mathbb{O})$ loop structure. Since m_{conf} is itself parameter-free (Addendum 106), μ_{match} is parameter-free at every level. The addendum also notes a typographic error in Addendum 127 equation (4), where $\pi(6 - \sqrt{3})/23$ appears in place of the correct $2\pi(6 - \sqrt{3})/23$; the abstract, numerical value, and all conclusions of Addendum 127 use the correct form and are unaffected.

1 Setup: the open item from Addendum 127

Addendum 127 resolved a convergence obstruction in the strong-coupling running series by exploiting a structural feature of the G_2 root system. The boundary condition $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$, established in Addendum 106 from the long-to-short root-length ratio of G_2 , places the expansion parameter far outside the perturbative window; fixed-order running from this point produces an asymptotic sequence that overshoots the PDG value $\alpha_s(M_Z) = 0.1179 \pm 0.0009$ by more than eleven standard deviations at $N^2\text{LO}$. Addendum 127 introduced a matching scale μ_{match} at which $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$, using the same G_2 angle $\theta = \pi/6$ that supplied the boundary condition, and derived

$$\mu_{\text{match}} = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right) \approx 219.99 \times 3.209 \approx 705.9 \text{ MeV}. \quad (1)$$

Matched perturbative running from this scale yields $\alpha_s^{\text{NLO}}(M_Z) = 0.1183 (+0.4\sigma)$ and $\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1176 (-0.3\sigma)$, a convergent bracket.

The exponent in equation (1) is exact and contains no free parameters, but Addendum 127 did not identify what the constituent factors 6, $\sqrt{3}$, and 23 represent in the TOE architecture. The abstract of Addendum 127 quotes the form $2\pi(6 - \sqrt{3})/23$ correctly. Equation (4) in the body of that addendum contains a typographic error, writing $\pi(6 - \sqrt{3})/23 \approx 1.166$ where the factor of 2π (not π) is required to reproduce the stated numerical value $219.99 \times 3.209 \approx 705.9 \text{ MeV}$; the factor $3.209 = e^{1.166}$ corresponds to the exponent $2\pi(6 - \sqrt{3})/23 \approx 1.1659$, not to $\pi(6 - \sqrt{3})/23 \approx 0.5830$. All conclusions of Addendum 127 follow from the correct form.

The gap to close is therefore an expository one: the exponent $2\pi(6 - \sqrt{3})/23$ is already parameter-free, but its TOE-native factorisation was not stated. The present addendum provides that factorisation and explains why each piece must appear.

2 Mechanism: G_2 root geometry and the $J_3(\mathbb{O})$ loop coefficient

The derivation in Addendum 127 proceeds by inverting the one-loop $\overline{\text{MS}}$ running equation,

$$\alpha_s(\mu) = \frac{\alpha_s(m_{\text{conf}})}{1 + \frac{b_0}{4\pi} \alpha_s(m_{\text{conf}}) \ln \frac{\mu^2}{m_{\text{conf}}^2}}, \quad b_0 = \frac{23}{3}, \quad (2)$$

with the two G_2 -derived boundary values $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$ and $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$. Setting $\alpha_s(\mu_{\text{match}}) = 1/2$ in equation (2) and solving for $\ln(\mu_{\text{match}}^2/m_{\text{conf}}^2)$ gives

$$\ln \frac{\mu_{\text{match}}^2}{m_{\text{conf}}^2} = (2\sqrt{3} - 1) \cdot \frac{12\pi}{23\sqrt{3}}. \quad (3)$$

Two algebraic steps reduce this to the compact form. First, the factor $(2\sqrt{3} - 1)/\sqrt{3}$ simplifies:

$$\frac{2\sqrt{3} - 1}{\sqrt{3}} = 2 - \frac{1}{\sqrt{3}} = \frac{2 \cdot 3 - \sqrt{3}}{3} = \frac{6 - \sqrt{3}}{3}. \quad (4)$$

Second, substituting into equation (3),

$$\ln \frac{\mu_{\text{match}}^2}{m_{\text{conf}}^2} = \frac{12\pi}{23} \cdot \frac{6 - \sqrt{3}}{3} = \frac{4\pi(6 - \sqrt{3})}{23}. \quad (5)$$

Halving to pass from $\ln(\mu^2/\mu_0^2)$ to $\ln(\mu/\mu_0)$,

$$\ln \frac{\mu_{\text{match}}}{m_{\text{conf}}} = \frac{2\pi(6 - \sqrt{3})}{23}. \quad (6)$$

The structural question is: what are the numbers 6, $\sqrt{3}$, and 23?

$\sqrt{3}$ and 6 from G_2 root geometry. The number $\sqrt{3}$ is the G_2 long-to-short root-length ratio $\rho_{G_2} = l/s$, first used in Addendum 106 to assign the boundary condition $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \rho_{G_2}$. The key observation is that equation (4) produces the integer 6 algebraically:

$$2\rho_{G_2}^2 - \rho_{G_2} = \rho_{G_2}(2\rho_{G_2} - 1) \implies 2 \cdot 3 - \sqrt{3} = 6 - \sqrt{3}, \quad (7)$$

so $6 - \sqrt{3} = \rho_{G_2}(2\rho_{G_2} - 1)$ with $\rho_{G_2} = \sqrt{3}$. The integer $6 = 2\rho_{G_2}^2$ is not arbitrary: it equals the number of positive roots of G_2 ,

$$|\Phi^+(G_2)| = 6 = 2\rho_{G_2}^2. \quad (8)$$

This is a combinatorial identity of the G_2 root system: G_2 has 12 roots in total (6 positive, 6 negative), and its root-length ratio $\rho_{G_2} = \sqrt{3}$ satisfies $2\rho_{G_2}^2 = 6 = |\Phi^+|$. The factor $(6 - \sqrt{3})$ in the exponent is therefore $|\Phi^+(G_2)| - \rho_{G_2}$, the positive-root count minus the root-length ratio.

23 from the $J_3(\mathbb{O})$ Casimir. The coefficient $b_0 = 23/3$ is the $n_f = 5$ one-loop $\overline{\text{MS}}$ coefficient. Addendum 121 derived it from the $J_3(\mathbb{O})$ exceptional Jordan algebra: the purely gluonic contribution $11C_A/3 = 11 \cdot 3/3 = 11$ and the quark-loop subtraction $-2n_f T_F/3 = -10/3$ combine to $b_0 = 23/3$ once $C_A = 3$, $T_F = 1/2$, $n_f = 5$ are taken from the $J_3(\mathbb{O})$ Casimir structure. The denominator $23 = 3b_0$ is therefore $J_3(\mathbb{O})$ -derived and carries no free parameters.

The prefactor 2π from the squared scale ratio. The factor 2 in front of π in equation (6) is a Jacobian: the one-loop running equation involves $\ln(\mu^2/m_{\text{conf}}^2)$, not $\ln(\mu/m_{\text{conf}})$. Converting from the logarithm of the squared ratio to the logarithm of the ratio introduces an exact factor of 2. In the TOE lapse formula $u = \cos(2\beta)$, the same doubling appears: the Hopf coordinate 2β winds twice around the S^1 fibre for each unit advance in β . The 2π in equation (6) is therefore the S^3 -geometric trace of the fact that renormalisation-group scales enter the one-loop kernel as squared ratios.

3 Derivation: the complete factorisation

Assembling the structural identifications above, the exponent in equation (6) reads

$$\frac{2\pi(6 - \sqrt{3})}{23} = \frac{2\pi \rho_{G_2}(2\rho_{G_2} - 1)}{3b_0} = \frac{2\pi(|\Phi^+(G_2)| - \rho_{G_2})}{3b_0}, \quad (9)$$

in which every symbol is a TOE-native quantity: π is the S^3 half-period, $\rho_{G_2} = \sqrt{3}$ is the G_2 root-length ratio, $|\Phi^+(G_2)| = 6$ is the positive-root count of G_2 , and $b_0 = 23/3$ is the $J_3(\mathbb{O})$ -derived one-loop coefficient. Exponentiating and multiplying by m_{conf} ,

$$\boxed{\mu_{\text{match}} = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(|\Phi^+(G_2)| - \rho_{G_2})}{3b_0}\right) = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right).} \quad (10)$$

Since $m_{\text{conf}} = \pi \alpha^{-1} m_e$ is itself parameter-free (Addendum 106, with $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ from the TOE), every quantity in equation (10) is derivable from the TOE without adjustment.

The derivation is internally consistent: the same G_2 angle $\theta = \pi/6$ that assigns the boundary value $\cot(\pi/6) = \rho_{G_2} = \sqrt{3}$ and the matching value $\sin(\pi/6) = 1/2$ also generates, through the algebraic simplification (4), the positive-root count $|\Phi^+(G_2)| = 2\rho_{G_2}^2 = 6$. Put differently, the number 6 is not inserted by hand — it is produced when $\cot$ and $\sin$ at the same angle $\pi/6$ are eliminated from the one-loop equation, leaving only G_2 root data.

The one-loop Landau pole provides a cross-check. Addendum 127 equation (6) gives

$$\Lambda_{\text{QCD}} = m_{\text{conf}} \cdot \exp\left(\frac{-6\pi}{23\sqrt{3}}\right) = \mu_{\text{match}} \cdot \exp\left(\frac{-12\pi}{23}\right) \approx 137.1 \text{ MeV}, \quad (11)$$

from which $\mu_{\text{match}}/\Lambda_{\text{QCD}} = \exp(12\pi/23)$. In the factorised language, $12\pi/23 = 4\pi\rho_{\text{G}_2}/(3b_0) \cdot \rho_{\text{G}_2}/\rho_{\text{G}_2}$ — again a G_2 -and- $J_3(\mathbb{O})$ expression — confirming that the two trajectories anchored at m_{conf} and μ_{match} share the same Landau pole through the same root geometry.

4 Verification

The numerical value follows by substituting $m_{\text{conf}} = \pi \alpha^{-1} m_e = \pi(4\pi^3 + \pi^2 + \pi)(0.511 \text{ MeV}) = 219.991 \text{ MeV}$,

$$\mu_{\text{match}} = 219.991 \times \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right) = 219.991 \times 3.2089 \approx 705.9 \text{ MeV}. \quad (12)$$

The exponent is $2\pi(6 - \sqrt{3})/23 = 1.16593$ and $e^{1.16593} = 3.2089$. This matches the value used throughout Addendum 127 exactly.

Factor	Value	TOE origin
m_{conf}	219.99 MeV	$\pi \alpha^{-1} m_e$, Addendum 106
2π	6.2832	S^3 squared-scale-ratio Jacobian / Hopf doubling
$6 = \Phi^+(\text{G}_2) $	6	positive-root count of G_2 ; $= 2\rho_{\text{G}_2}^2$
$\sqrt{3} = \rho_{\text{G}_2}$	1.7321	G_2 long/short root-length ratio
$23 = 3b_0$	23	$J_3(\mathbb{O})$ Casimir, $n_f = 5$ one-loop, Addendum 121
μ_{match}	705.9 MeV	fully derived; zero free parameters

Table 1: Complete structural provenance of the matching scale. Every factor in equation (10) is TOE-native.

It is worth comparing with the numerical near-miss $\mu_{\text{match}} \approx m_{\text{conf}} \cdot 2\phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. One finds $2\phi \approx 3.2361$ against $e^{2\pi(6 - \sqrt{3})/23} \approx 3.2089$, a discrepancy of +0.85%. This proximity is a numerical coincidence: the golden ratio does not participate in G_2 root geometry or the $J_3(\mathbb{O})$ Casimir structure in this context, and no structural derivation leading from ϕ to the exponent $2\pi(6 - \sqrt{3})/23$ is available within the framework. The 0.85% gap between 2ϕ and the exact result is not a residual to be closed; it is the signature of a non-structural near-miss. The structural derivation is equation (9), and it closes at zero residual by construction.

5 Open items

Uniqueness of the G_2 angle. The derivation relies on both boundary and matching values being assigned by the single angle $\theta = \pi/6$ of the G_2 root fan. A more complete treatment would derive from the TOE why this particular angle in this particular exceptional group governs the non-perturbative boundary of QCD, rather than taking it as the natural identification. The relevant structure is the embedding $\text{G}_2 \subset \text{Spin}(7)$ and the role of G_2 as the automorphism group of the octonions $\mathbb{O}$ that enter $J_3(\mathbb{O})$; the angle $\pi/6$ then arises from the half-opening of the root fan under the $\text{SU}(3)$ subgroup that is identified with QCD colour. This sub-derivation would make the chain of inference fully closed.

b_0 at higher flavour thresholds. The denominator $23 = 3b_0$ is the $n_f = 5$ value. Between m_{conf} and μ_{match} , thresholds for the strange and charm quarks are crossed. Addendum 127

uses $n_f = 5$ throughout; a more precise treatment would run with $n_f = 3$ from m_{conf} to $m_c \approx 1.27$ GeV, and with $n_f = 4$ above m_c . Since $\mu_{\text{match}} \approx 706$ MeV lies below m_c and above the strange threshold $m_s \approx 95$ MeV, the natural choice is $n_f = 3$ ($b_0 = 9$) or $n_f = 4$ ($b_0 = 25/3$) for the running in this region. Whether the G_2 matching condition $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$ is preserved under this threshold correction, or whether it shifts μ_{match} , is an open calculation.

The Hopf-lapse interpretation. The factor 2π in the exponent is attributed here to the squared-scale Jacobian. A more precise statement within the TOE would identify the Hopf latitude $u = \cos(2\beta)$ corresponding to μ_{match} on the S^3 trajectory, and verify that the double-winding interpretation is consistent with the GON dynamics at this scale. This would connect the matching condition to a specific locus on S^3 and give μ_{match} a geometric rather than merely algebraic status.