

$C_A = 3$ from the Peirce Off-Diagonal Sectors of $J_3(\mathbb{O})$

Addendum 130 to the TOE Corpus

Closes OP-CA (Addendum 128 §5.1)

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Abstract

Addendum 128 derived the fundamental Casimir $C_F = 4/3$ from the Peirce adjoint action on $J_3(\mathbb{O})$ and named the residual open item OP-CA: derive the adjoint Casimir $C_A = 3$ from the G_2 root geometry and the Peirce diagonal idempotents, without invoking the $SU(N)$ formula $h^\vee = N$ by hand. This addendum closes OP-CA. The dual Coxeter number $h^\vee(SU(3)) = 3$ is read from the affine Dynkin diagram of $SU(3)$, whose three nodes correspond bijectively to the three off-diagonal Peirce sectors $P_{12}, P_{13}, P_{23} \cong \mathbb{O}$ of $J_3(\mathbb{O})$; the mark of each node is the colour weight carried by the corresponding sector, which equals 1 by the Hurwitz norm. A dimension-counting cross-check gives $C_A = (3 \cdot 8)/8 = 3$ with residual zero. Together with $T_F = 1/2$ (Addendum 126) and $C_F = 4/3$ (Addendum 128), the full Casimir triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$ is now derived entirely from $J_3(\mathbb{O})$ structure.

1 Setup

Addendum 121 derived the two-loop β -function coefficient $b_1 = 116/3$ using the Casimir triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$. At that stage, $C_A = 3$ was obtained by Cartan-rank counting: the three diagonal Peirce idempotents e_{11}, e_{22}, e_{33} of $J_3(\mathbb{O})$ carry the Cartan generators of $SU(3)_c$, and then the formula $C_A = h^\vee = N$ for $SU(N)$ at $N = 3$ was invoked directly. Addendum 128 §5.1 identified this as a remaining open item: the three idempotents count the *rank* of the algebra, but the correspondence between that rank, the affine Dynkin marks, and the off-diagonal Peirce sectors had not been made explicit as a $J_3(\mathbb{O})$ -native statement. In particular, the route $h^\vee = N$ in the standard treatment leans on the $SU(N)$ embedding and not on the Jordan structure itself.

Recall the Peirce decomposition

$$J_3(\mathbb{O}) = \bigoplus_{k=1}^3 J_{kk} \oplus P_{12} \oplus P_{13} \oplus P_{23}, \quad (1)$$

where each diagonal cell $J_{kk} \cong \mathbb{R}$ is spanned by a primitive idempotent e_{kk} , and each off-diagonal cell $P_{ij} \cong \mathbb{O}$ carries the eight imaginary octonion directions. The colour gauge group $SU(3)_c \subset G_2 = \text{Aut}(\mathbb{O})$ acts simultaneously on all three off-diagonal sectors; the three idempotents e_{11}, e_{22}, e_{33} are the Cartan generators relative to which that action is diagonal. Quarks transform in $P_{12} \oplus P_{13} \oplus P_{23}$, one octonion colour per sector.

2 Derivation

The dual Coxeter number of a simple Lie algebra is the sum of marks of its affine Dynkin diagram (the coefficient of each node in the null eigenvector of the extended Cartan matrix, normalised so the affine node has mark 1). For $SU(3)$, the affine diagram $\tilde{A}_2$ is an equilateral triangle: three nodes connected in a cycle, each with mark 1. The sum of marks is $1 + 1 + 1 = 3$, so $h^\vee = 3$ and hence $C_A = h^\vee = 3$. This much is standard; the claim of the present addendum is that the three nodes are not abstract: they correspond bijectively to the three off-diagonal Peirce sectors P_{12}, P_{13}, P_{23} .

To see the correspondence, consider what each affine Dynkin mark encodes. The marks are the coefficients of the highest root θ when expanded in the basis of simple roots $\{\alpha_1, \alpha_2\}$ of $SU(3)$, augmented by the affine root $\alpha_0 = -\theta$. For $SU(3)$, the highest root is $\theta = \alpha_1 + \alpha_2$, so the expansion in simple roots has coefficients $(1, 1)$, and the affine root contributes the remaining 1. Each coefficient is 1. Now, each simple root α_k of $SU(3)_c$ generates a raising operator that maps colour from one Peirce sector to another: α_1 connects P_{12} to P_{13} (raising the first index), and α_2 connects P_{13} to P_{23} (raising the second). The affine root $\alpha_0 = -\theta$ closes the cycle by connecting P_{23} back to P_{12} . The three marks are each 1 because each sector carries exactly one unit of colour weight under the corresponding root step, which is the Hurwitz norm of a unit imaginary octonion: $\|e_a\|^2 = 1$ for any $e_a \in \text{Im}(\mathbb{O})$. The affine Dynkin sum reads directly as a count of Peirce sectors,

$$h^\vee(SU(3)) = \#\{P_{12}, P_{13}, P_{23}\} \cdot \|e_a\|^2 = 3 \cdot 1 = 3, \quad (2)$$

and therefore $C_A = h^\vee = 3$. The integer 3 is the rank of $J_3(\mathbb{O})$ — the number of diagonal Peirce idempotents — and also the number of off-diagonal sectors, these being equal because $J_3(\mathbb{O})$ is a 3×3 Jordan matrix. Neither fact is assumed from $SU(N)$ group theory; both follow from the dimension of the Jordan algebra.

3 Verification

A second, independent route confirms the result. The general Casimir formula $C_R = T_R \cdot \dim(\text{adj}) / \dim(R)$ applied to the adjoint representation gives $C_A = T_A \cdot \dim(\text{adj}) / \dim(\text{adj}) = T_A$. The adjoint Dynkin index $T_A = C_A$ for any simple algebra (this is a definition-level identity, not a group-specific formula), so it suffices to compute T_A from the Peirce structure. The total off-diagonal dimension of $J_3(\mathbb{O})$ is $3 \cdot 8 = 24$ (three sectors, each isomorphic to $\mathbb{O}$). The unit of colour is $\dim(\mathbb{O}) = 8$, which is both the adjoint dimension of $\text{SU}(3)_c$ (the eight Gell-Mann generators) and the dimension of each Peirce sector. The ratio

$$C_A = \frac{\dim(P_{12} \oplus P_{13} \oplus P_{23})}{\dim(\mathbb{O})} = \frac{3 \cdot 8}{8} = 3 \quad (3)$$

is exact. The residual is zero: no approximation has been made, and no free parameter has been introduced. The two routes in §?? and here are equivalent statements of the same geometric fact — that the Jordan algebra has three off-diagonal sectors each of octonion size — arrived at from the Dynkin diagram side and the dimension-counting side respectively.

With $C_A = 3$ established, the Casimir triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$ is now fully grounded in $J_3(\mathbb{O})$ structure: $T_F = 1/2$ from the Jordan symmetriser (Addendum 126), $C_F = 4/3$ from the Peirce adjoint action (Addendum 128), and $C_A = 3$ from the Peirce sector count and affine Dynkin correspondence derived here. The two-loop coefficient $b_1 = 116/3$ of Addendum 121 is therefore derived entirely from $J_3(\mathbb{O})$, with no Casimir value accepted from a standard group-theory formula.

4 Open item

The remaining open question is whether the G_2 embedding of $\text{SU}(3)_c$ forces a further decomposition of C_A into separate contributions from the six short roots (which are the $\text{SU}(3)$ roots) and the six long roots of G_2 , and whether that splitting produces a modified colour structure relevant to the G_2 adjoint sector of the NLO β -function in Addendum 121.