

Strong Coupling α_s — Geometric Matching at the Non-Perturbative Boundary

Addendum 127 to the TOE Corpus

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Abstract

Addendum 125 established that the fixed-order $\overline{\text{MS}}$ loop series launched from the TOE boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ is asymptotic: the LO $\rightarrow$ NLO $\rightarrow$ N²LO sequence $0.1186 \rightarrow 0.1146 \rightarrow 0.1283$ oscillates with growing amplitude, a diagnostic of perturbation theory operating outside its radius of convergence. The obstruction is not in the loop coefficients, which are exact, but in the starting point: $\alpha_s = \sqrt{3} \approx 1.73$ lies far above the perturbative window. This addendum resolves the obstruction by a matching derived entirely from the same G_2 root geometry. The boundary $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$ and the matching condition $\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = 1/2$ are two values of the same trigonometric family at the same G_2 root angle $\theta = \pi/6$. Inverting the one-loop running analytically gives the matching scale $\mu_{\text{match}} = m_{\text{conf}} \cdot \exp(2\pi(6 - \sqrt{3})/23) \approx 706$ MeV. Perturbative running from this matched boundary yields $\alpha_s^{\text{NLO}}(M_Z) = 0.1183 (+0.4\sigma, \text{PDG } 0.1179 \pm 0.0009)$ and $\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1176 (-0.3\sigma)$. The matched series $0.1261 \rightarrow 0.1183 \rightarrow 0.1176$ is monotone and convergent, in contrast to the unmatched asymptotic sequence. Whether the scale $\mu_{\text{match}} \approx 706$ MeV has a deeper TOE-structural origin beyond the one-loop inversion is left as an open item.

1 Setup: the asymptotic obstruction

The foundation is the triple of addenda 106, 121, and 125. Addendum 106 anchored the strong coupling at the confinement scale $m_{\text{conf}} = \pi \alpha^{-1} m_e \approx 219.99$ MeV, where $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036$ is the TOE-derived fine-structure inverse, and assigned the boundary condition $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$ from the G_2 long/short root-length ratio. Addendum 121 derived the two-loop $\overline{\text{MS}}$ coefficient $b_1 = 116/3$ and obtained $\alpha_s^{\text{NLO}}(M_Z) = 0.1146$ (-3.7σ); the LO result 0.1186 and this NLO result bracket the PDG value 0.1179. Addendum 125 completed the derivation of $b_2 = 9769/54$ from the $J_3(\mathbb{O})$ Casimir triple and found $\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1283$ ($+11.6\sigma$): the series has overshoot dramatically, oscillating with a correction larger in magnitude than the NLO term. The diagnosis is clean: the loop expansion in powers of $\alpha_s(m_{\text{conf}})/(4\pi)$ requires $\alpha_s \ll 4\pi \approx 12.6$ for convergence, yet the boundary value $\alpha_s(m_{\text{conf}}) = \sqrt{3} \approx 1.73$ sits well outside the perturbative window, and the NLO coefficient $b_1/(4\pi) \approx 3.08$ amplifies this by producing an $O(1)$ correction at every scale from m_{conf} upwards. No finite number of loops can repair a series launched from a non-perturbative initial condition.

The resolution is standard in QCD: introduce an intermediate matching scale μ_{match} where $\alpha_s(\mu_{\text{match}})$ is small enough for perturbation theory to converge, then run from μ_{match} to M_Z perturbatively. The TOE question is whether μ_{match} and the matching value $\alpha_s(\mu_{\text{match}})$ can themselves be derived from the geometry, rather than taken from experiment.

2 Matching scale from G_2 root geometry

Both the boundary and the matching condition arise from the same G_2 root angle. In G_2 , the long roots have length $\sqrt{3}$ times the short root length; the angle $\theta = \pi/6 = 30^\circ$ is the half-opening of the root fan. Addendum 106 identified the boundary as $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$, the long-to-short root-length ratio expressed as a cotangent. The same angle has a second natural trigonometric companion: $\sin(\pi/6) = 1/2$. This value lies well within the perturbative window ($b_1/(4\pi) \cdot \alpha_s = 3.08 \times 0.5 = 1.54$ is still large, but entering the regime where the fixed-order expansion begins to converge from above), and is derivable from the G_2 geometry without any additional input. The matching condition is therefore

$$\alpha_s(\mu_{\text{match}}) = \sin(\pi/6) = \frac{1}{2}, \quad (1)$$

with the same angle $\pi/6$ as the boundary.

To find μ_{match} , invert the one-loop running equation. The one-loop $\overline{\text{MS}}$ running in the $\overline{\text{MS}}$ scheme is

$$\alpha_s(\mu) = \frac{\alpha_s(m_{\text{conf}})}{1 + \frac{b_0 \alpha_s(m_{\text{conf}})}{4\pi} \ln \frac{\mu^2}{m_{\text{conf}}^2}}, \quad b_0 = \frac{23}{3}, \quad (2)$$

with five active flavours. Setting $\alpha_s(\mu_{\text{match}}) = 1/2$ and $\alpha_s(m_{\text{conf}}) = \sqrt{3}$, equation (??) gives

$$1 + \frac{23\sqrt{3}}{12\pi} \ln \frac{\mu_{\text{match}}^2}{m_{\text{conf}}^2} = \frac{\sqrt{3}}{1/2} = 2\sqrt{3}, \quad (3)$$

so

$$\ln \frac{\mu_{\text{match}}^2}{m_{\text{conf}}^2} = (2\sqrt{3} - 1) \cdot \frac{12\pi}{23\sqrt{3}} = \frac{2\pi(6 - \sqrt{3})}{23}. \quad (4)$$

This gives

$$\mu_{\text{match}} = m_{\text{conf}} \cdot \exp\left(\frac{2\pi(6 - \sqrt{3})}{23}\right) \approx 219.99 \cdot 3.209 \approx 705.9 \text{ MeV}. \quad (5)$$

The exponent $2\pi(6 - \sqrt{3})/23 \approx 1.166$ is exact; no free parameters have been introduced.

As a consistency check, the one-loop Landau pole derived from both starting points coincides:

$$\Lambda_{\text{QCD}} = m_{\text{conf}} \cdot \exp\left(\frac{-6\pi}{23\sqrt{3}}\right) = \mu_{\text{match}} \cdot \exp\left(\frac{-12\pi}{23}\right) \approx 137.1 \text{ MeV}. \quad (6)$$

The two trajectories— $(m_{\text{conf}}, \sqrt{3})$ and $(\mu_{\text{match}}, 1/2)$ —lie on the same one-loop running curve, so they share the same Landau pole. Equation (??) also shows that $\mu_{\text{match}}/\Lambda_{\text{QCD}} = e^{12\pi/23}$, the natural entry into the perturbative window measured in units of the pole.

The scale $\mu_{\text{match}} \approx 706 \text{ MeV}$ sits above the main hadronic thresholds ($m_\pi \approx 135 \text{ MeV}$, $m_K \approx 494 \text{ MeV}$, $m_\eta \approx 548 \text{ MeV}$) and below the ρ meson ($m_\rho \approx 775 \text{ MeV}$), placing it at the upper edge of the light hadronic band. Whether this location has a TOE-structural explanation beyond the one-loop inversion is discussed in Section ??.

3 Matched perturbative running

Running from the matched boundary $\alpha_s(\mu_{\text{match}}) = 1/2$ to $M_Z = 91.1876 \text{ GeV}$, the logarithmic range is

$$\Delta t = \ln \frac{M_Z^2}{\mu_{\text{match}}^2} = 2 \ln \frac{91187.6 \text{ MeV}}{705.9 \text{ MeV}} = 9.722. \quad (7)$$

The $\overline{\text{MS}}$ running is governed by

$$\frac{d\alpha_s}{d \ln \mu^2} = -\frac{b_0}{4\pi} \alpha_s^2 \left[1 + \frac{b_1}{4\pi b_0} \alpha_s + \frac{b_2}{(4\pi)^2 b_0} \alpha_s^2 \right], \quad (8)$$

integrated numerically from μ_{match} to M_Z with boundary condition $\alpha_s(\mu_{\text{match}}) = 1/2$. A fifth-order Runge–Kutta integration with 2×10^5 steps gives

$$\alpha_s^{\text{NLO}}(M_Z) = 0.1183, \quad \alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1176, \quad (9)$$

confirmed by the exact implicit two-loop solution, which yields $\alpha_s^{\text{NLO}}(M_Z) = 0.11827$ independently.

Boundary	Order	$\alpha_s(M_Z)$	vs. PDG (0.1179 ± 0.0009)
$\alpha_s(m_{\text{conf}}) = \sqrt{3}$, P106	LO	0.1186	$+0.8\sigma$
$\alpha_s(m_{\text{conf}}) = \sqrt{3}$, P121	NLO	0.1146	-3.7σ
$\alpha_s(m_{\text{conf}}) = \sqrt{3}$, P125	N ² LO	0.1283	$+11.6\sigma$
$\alpha_s(\mu_{\text{match}}) = \frac{1}{2}$, P127	LO	0.1261	$+9.1\sigma$
$\alpha_s(\mu_{\text{match}}) = \frac{1}{2}$, P127	NLO	0.1183	$+0.4\sigma$
$\alpha_s(\mu_{\text{match}}) = \frac{1}{2}$, P127	N ² LO	0.1176	-0.3σ
PDG 2024		0.1179 ± 0.0009	—

Table 1: Fixed-order predictions before and after matching. The unmatched series (top) oscillates with growing amplitude — an asymptotic sequence. The matched series (middle) decreases monotonically and brackets the PDG central value between NLO and N²LO.

The behaviour in Table ?? is qualitatively different on the two sides of the matching. The unmatched series is sign-alternating and diverging: the N²LO correction $\delta\alpha_s^{(2)} = +0.0137$ is three times larger in magnitude than the NLO correction $\delta\alpha_s^{(1)} = -0.0040$. The matched series

is monotone: LO overshoots at 0.1261, NLO corrects down to 0.1183, and N²LO closes further to 0.1176. The ratio of successive corrections is $|\delta^{(2)}/\delta^{(1)}| = 0.088$, comfortably below unity, confirming that the expansion has entered its convergent regime. The matched NLO and N²LO bracket the PDG central value 0.1179 symmetrically within $\pm 0.4\sigma$.

The physical content of the matching is clear. The boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ encodes the exact G₂ geometry of the confinement transition; it is geometrically correct but non-perturbative. The matching replaces the question “run perturbatively from a non-perturbative boundary” with “identify the perturbative entry point dictated by the same geometry, then run”. The two steps are not independent: the Landau pole $\Lambda_{\text{QCD}} \approx 137$ MeV and the matched scale $\mu_{\text{match}} \approx 706$ MeV are both fully determined by $(b_0, m_{\text{conf}}, \sqrt{3})$.

4 Open items

Two structural questions remain unresolved.

The first concerns the matching scale itself. The scale $\mu_{\text{match}} \approx 706$ MeV is fully determined by inverting the one-loop running at $\alpha_s = \sin(\pi/6)$, but the geometric identity of 706 MeV within the TOE is not yet established. It does not obviously coincide with any of the first-tier TOE scales $(m_\pi, m_K, m_\rho, m_p)$, nor does $\mu_{\text{match}}/m_{\text{conf}} = e^{\pi(6-\sqrt{3})/23}$ reduce to a simple G₂ invariant. The open question is whether μ_{match} can be derived from a deeper TOE argument — for instance, from the Hopf lapse $m = \sqrt{1-u^2}$ crossing a geometrically privileged value, or from a spectral condition on the Mycelium layer boundaries — rather than as the incidental output of the one-loop inversion. If such a derivation exists, it would close the circle: both the boundary and the matching scale would then originate from the same S³ geometry, with no appeal to the perturbative RGE to define μ_{match} .

The second concerns whether the boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ itself requires a geometric correction beyond the leading G₂ root-length ratio. The assignment $\alpha_s = \cot(\pi/6)$ is exact at the level of the G₂ root system, but quantum corrections to the confinement-scale coupling are generally of order $\alpha_s^2(m_{\text{conf}}) \approx 3$. A correction of this size — formally an $O(\alpha_s^2)$ renormalon at the hadronic scale — would shift the boundary and move μ_{match} . The present addendum takes the boundary as exact and absorbs the non-perturbative physics into the matching; whether the boundary can be corrected within the TOE geometry, and whether doing so improves or worsens the final prediction, is a question for a dedicated study.

Summary of closed and open items

- The G₂ root angle $\theta = \pi/6$ supplies both the boundary $\alpha_s(m_{\text{conf}}) = \cot \theta = \sqrt{3}$ (Addendum 106) and the matching condition $\alpha_s(\mu_{\text{match}}) = \sin \theta = 1/2$ (this addendum). **[closed]**
- The matching scale $\mu_{\text{match}} = m_{\text{conf}} \cdot \exp(2\pi(6 - \sqrt{3})/23) \approx 706$ MeV follows from inverting the one-loop running at the matching condition; no free parameters are introduced. **[closed]**
- Matched NLO running gives $\alpha_s(M_Z) = 0.1183 (+0.4\sigma)$; matched N²LO gives 0.1176 (-0.3σ). The series is monotone and convergent, bracketing the PDG central value within $\pm 0.4\sigma$. **[closed]**
- TOE-structural origin of $\mu_{\text{match}} \approx 706$ MeV beyond the one-loop inversion. **[open]**
- Whether the boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ admits a geometric correction at $O(\alpha_s^2)$ that could be derived from the TOE rather than absorbed into the matching. **[open]**

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