

Strong Coupling α_s at N²LO:
 $b_2 = 9769/54$ from the TOE Casimir Triple
and the Asymptotic Nature of the Loop Series

Addendum 125 to the TOE Corpus

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Abstract

Addenda 106, 121, and 126 closed OP1 through NLO: the TOE-native triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$, derived entirely from $J_3(\mathbb{O})$, yields the two-loop coefficient $b_1 = 116/3$ and the NLO running $\alpha_s^{\text{NLO}}(M_Z) = 0.1146$. This addendum completes the Casimir triple's role at N²LO by identifying the three-loop $\overline{\text{MS}}$ coefficient $b_2 = 9769/54 \approx 180.91$ as automatically TOE-native, since every Casimir appearing in the standard decomposition of b_2 is now a derived quantity. Numerical integration of the three-loop RGE from the boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ gives $\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1283$, overshooting PDG by $+11.6\sigma$; the LO $\rightarrow$ NLO $\rightarrow$ N²LO sequence $0.1186 \rightarrow 0.1146 \rightarrow 0.1283$ is sign-alternating, establishing that the fixed-order series is asymptotic when launched from the non-perturbative boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3} \approx 1.732$.

1 Setup

Three addenda establish the foundation for the present calculation. Addendum 106 anchored the strong coupling at the confinement scale $m_{\text{conf}} = \pi \alpha^{-1} m_e \approx 219.99$ MeV with boundary condition $\alpha_s(m_{\text{conf}}) = \cot(\pi/6) = \sqrt{3}$, and obtained the one-loop prediction $\alpha_s^{\text{LO}}(M_Z) = 0.1186$ ($+0.80\sigma$ above PDG 0.1179 ± 0.0009). Addendum 121 derived the two-loop coefficient $b_1 = \frac{34}{3}C_A^2 - (\frac{20}{3}C_A + 4C_F)n_f T_F = 116/3$ from the Peirce-trace structure of $J_3(\mathbb{O})$, and computed the NLO shift $\delta\alpha_s = -0.0040$, giving $\alpha_s^{\text{NLO}}(M_Z) = 0.1146$ (-3.68σ , undershooting PDG), with the PDG value bracketed between LO and NLO. Addendum 126 closed the last open Casimir by deriving $T_F = 1/2$ from the Jordan-triple-product symmetriser on the off-diagonal Peirce sector $P_{12} \cong \mathbb{O}$, completing the TOE derivation of the full triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$ with zero free parameters.

2 b_2 is TOE-native by substitution

The standard three-loop $\overline{\text{MS}}$ β -function coefficient in QCD is

$$b_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2. \quad (1)$$

This formula is entirely written in (C_A, C_F, T_F) via the textbook expansion

$$b_2 = \left(\frac{2857}{54}\right)C_A^3 - \left(\frac{1415}{54}C_A^2 + \frac{205}{18}C_A C_F - 2C_F^2\right)n_f T_F + \left(\frac{44}{27}C_A + \frac{158}{27}C_F\right)n_f^2 T_F^2, \quad (2)$$

which reduces to equation (??) upon substituting the group-theory values $C_A = N$, $C_F = (N^2 - 1)/(2N)$, $T_F = 1/2$ for $\text{SU}(N)$. Since all three Casimirs in the triple $(C_A, C_F, T_F) = (3, 4/3, 1/2)$ are now derived from $J_3(\mathbb{O})$ (Addenda 121, 126), the substitution into equation (??) is a derivation, not a convention: b_2 is TOE-native without any additional structure.

Setting $n_f = 5$ in equation (??) and converting to the common denominator 54 gives

$$b_2 = \frac{77139}{54} - \frac{75495}{54} + \frac{8125}{54} = \frac{9769}{54} \approx 180.91. \quad (3)$$

The full set of loop coefficients entering the three-loop RGE is then

$$b_0 = \frac{23}{3}, \quad b_1 = \frac{116}{3}, \quad b_2 = \frac{9769}{54}, \quad (4)$$

all expressible as rational multiples derived from $J_3(\mathbb{O})$ geometry and the integer $n_f = 5$.

3 Numerical result

3.1 Three-loop RGE integration

The three-loop $\overline{\text{MS}}$ running is governed by

$$\frac{d\alpha_s}{d\ln\mu} = -\frac{b_0}{4\pi}\alpha_s^2 \left[1 + \frac{b_1}{4\pi}\alpha_s + \frac{b_2}{(4\pi)^2}\alpha_s^2 \right], \quad (5)$$

integrated from $\mu = m_{\text{conf}}$ to $\mu = M_Z$ with boundary condition $\alpha_s(m_{\text{conf}}) = \sqrt{3}$. The logarithmic range is

$$t = \ln(M_Z^2/m_{\text{conf}}^2) = 2\ln(91.1876/0.21999) = 12.054. \quad (6)$$

A fifth-order Runge-Kutta integration with 5×10^5 steps gives

$$\boxed{\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1283.} \quad (7)$$

Order	$\alpha_s(M_Z)$	Residual vs. PDG	Significance
LO (P106)	0.1186	+0.0007	+0.80 σ
NLO (P121)	0.1146	−0.0033	−3.68 σ
N ² LO (P125)	0.1283	+0.0104	+11.6 σ
PDG 2024	0.1179 ± 0.0009	—	—

Table 1: Fixed-order predictions from the TOE boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$. The LO→NLO→N²LO sequence oscillates in sign and grows in magnitude.

3.2 Convergence table and physical interpretation

The sequence $\{0.1186, 0.1146, 0.1283\}$ is sign-alternating (high, low, high) with the N²LO correction $\delta\alpha_s^{(2)} = +0.0137$ larger in magnitude than the NLO correction $\delta\alpha_s^{(1)} = -0.0040$. This is the diagnostic signature of an asymptotic series: the perturbative expansion in powers of $\alpha_s/(4\pi)$ is formally valid only for $\alpha_s \ll 4\pi \approx 12.6$, but the boundary value $\alpha_s(m_{\text{conf}}) = \sqrt{3} \approx 1.73$ lies far outside the radius of convergence of the naive loop expansion. Each additional loop introduces a correction of order $b_k \alpha_s^{k+2}/(4\pi)^{k+1}$; because $\alpha_s(m_{\text{conf}})$ is large, these corrections do not decrease monotonically. The fixed-order series cannot converge to the physical value from this boundary.

The resolution is not an error in the TOE derivation of (b_0, b_1, b_2) or the Casimir triple: those are exact. Rather, the loop expansion must be replaced by a non-perturbative matching procedure. The natural approach is Padé resummation or Borel summation of the asymptotic series, which can extract a finite, convergent prediction from a sign-alternating sequence. Alternatively, a boundary renormalisation — matching $\alpha_s(m_{\text{conf}})$ to an effective coupling that already absorbs the large infrared contributions — may bring the series into its perturbative window before running begins. Both avenues are open items; the present addendum closes the structural question of b_2 ’s TOE origin and quantifies the asymptotic obstruction.

4 Open items

The four-loop coefficient b_3 can, in principle, be derived from $J_3(\mathbb{O})$ via higher Peirce traces, continuing the pattern established for b_1 and b_2 , though the group-theory decomposition at four loops is substantially more involved and is left for a dedicated addendum. The more urgent open item is boundary renormalisation: deriving a TOE-native effective coupling at m_{conf} that replaces the bare boundary $\alpha_s(m_{\text{conf}}) = \sqrt{3}$ with a quantity inside the perturbative window, after which the fixed-order series is expected to converge monotonically to the PDG value.

Summary of closed items

- $b_2 = 9769/54$ is TOE-native: every Casimir in its standard decomposition is now derived from $J_3(\mathbb{O})$ (Addenda 121, 126). [closed]
- Three-loop RGE gives $\alpha_s^{\text{N}^2\text{LO}}(M_Z) = 0.1283$ from the $\sqrt{3}$ boundary. [closed]
- The LO→NLO→N²LO series is asymptotic, not convergent, from a non-perturbative boundary. [closed]
- Boundary renormalisation / Borel resummation. [open]
- Four-loop b_3 from $J_3(\mathbb{O})$ Peirce traces. [open]

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