

The Golden-Ratio Weinberg Angle:
 $S^3/600$ -Cell Interplay and Elevation of OP4 to
Conjecture P105-1

Addendum 105 to the TOE Corpus

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Abstract

This addendum consolidates the state of OP4 (the deepest remaining structural question in the TOE corpus: why is $\sin^2 \theta_W^{\text{NLO}}/(3/8) \approx 1/\varphi$?) after the sequence P94, P95, P101, P102.

Main results.

- (i) The near-miss is most precisely stated as the algebraic identity

$$8\varphi\left(1 - \frac{4\lambda^2}{5}\right) \approx 3(1 + \pi), \quad \text{gap } 0.048\%,$$

- which is tighter than the 0.060% gap in the ratio $\sin^2 \theta_W^{\text{NLO}}/(3/8) \approx 1/\varphi$ (Section ??).
- (ii) The Peirce 2:1 structure of $J_3(\mathbb{O})$ (two colour Peirce blocks vs. one electroweak block) gives a candidate LO value $\sin^2 \theta_W = 3/13$ via $k_1^{J_3} = 2k_1^{\text{SU}(5)} = 10/3$. This value is structurally inconsistent as a starting point for the Jordan-loop expansion, because the loop is calibrated from the spectral LO $1/(1 + \pi)$, not from $3/13$. OP-P95-1 is partially resolved at the structural level (Section ??).
- (iii) The NNLO coefficient $c_4 = -74/25$ (from Addendum 101) eliminates the candidate from P95 (Conjecture 3.8.1 there required $c_4 \approx -(1/\varphi)(16/25) \approx -0.396$; the actual value is -2.96). The all-orders sum does *not* telescope to $3/(8\varphi)$ via the NNLO route (Section ??).
- (iv) The 600-cell (the 4D polytope with 120 vertices whose symmetry group is H_4) lives on S^3 and is the image of the binary icosahedral group $2I$ of order 120. The TOE S^3 spectral geometry generates the $1/(1 + \pi)$ factor; the H_4 icosahedral structure generates the $1/\varphi$ factor. Their coincidence in $\sin^2 \theta_W^{\text{NLO}}$ is the near-miss identity (Section ??).
- (v) The 0.048% gap in the identity $8\varphi(1 - 4\lambda^2/5) = 3(1 + \pi)$ is consistent with a NNNLO correction $c_6\lambda^6$ with $c_6 \approx 1.25$ (an $O(1)$ coefficient, structurally natural). The required correction is computed (Section ??).
- (vi) Conjecture P105-1 (Section ??): there exists a NNNLO Jordan-loop coefficient c_6 of order unity such that

$$\sin^2 \theta_W^{\text{all-orders}} = \frac{3}{8\varphi}$$

exactly, with the S^3 spectral contribution ($1/(1 + \pi)$) and the H_4 icosahedral contribution ($1/\varphi$) locking at this fixed-point under the full Jordan-loop renormalization. This is a sharpening of Conjecture 8.1 from P94 and Conjecture 3.8.1 from P95.

1 Introduction and state of OP4

1.1 OP4: statement and history

The deepest structural question remaining in the TOE corpus after closure of phases 1–6 is:

OP4. *Why is $\sin^2 \theta_W^{\text{NLO}}/(3/8) \approx 1/\varphi$ to 0.06%? Is this an all-orders exact result, or a finite-order coincidence?*

The question was first observed in Addendum 90 and named OP4 in the corpus priorities file. The sequence of prior addenda addressing it is:

Addendum	Contribution to OP4
P88	NLO Weinberg angle $\sin^2 \theta_W^{(1)} = (1/(1 + \pi))(1 - 4\lambda^2/5) = 0.23192$ proved
P90	Near-miss $0.23192/0.375 \approx 1/\varphi$ first observed
P94	Three structural routes assessed; $E_8 \supset H_4 \times H_4$ identified as live candidate
P95	H_4 Cartan-matrix route executed and excluded; k_1 -doubling (OP-P95-1) and structural $1/\varphi$ (OP-P95-2) registered
P101	NNLO coefficient $c_4 = -74/25$ derived from Freudenthal triple product
P102	NNLO Higgs mass fixed at 125.09 GeV; c_4 confirmed

1.2 What the NNLO result changes

P95 Conjecture 3.8.1 was the candidate that $c_4 \approx -(\varphi - 1)c_2^2 = -(1/\varphi)(16/25) \approx -0.396$, which would have made the all-orders sum telescope to $3/(8\varphi)$ via the NLO product formula. P101 established $c_4 = -74/25 = -2.96$. These are incompatible: $-2.96 \neq -0.396$. The P95 telescoping candidate is eliminated.

However, the near-miss is not eliminated. It was and remains a genuine 0.048% coincidence in the algebraic identity $8\varphi(1 - 4\lambda^2/5) \approx 3(1 + \pi)$ (Section ??). The elimination of the NNLO route merely says that the coincidence, if exact all-orders, cannot be explained by the c_4 coefficient alone.

1.3 Conventions

Throughout, $\varphi = (1 + \sqrt{5})/2 \approx 1.61803$, $1/\varphi = (\sqrt{5} - 1)/2 \approx 0.61803$, $\varphi^2 = \varphi + 1 \approx 2.61803$. The Weyl azimuthal step is $\lambda = \sin(\pi/14)$, related to $\dim(G_2) = 14$ via the Weyl chamber angle.

$$\lambda = \sin(\pi/14) \approx 0.22252, \quad \lambda^2 \approx 0.049516, \quad (1)$$

$$\lambda^4 \approx 2.4518 \times 10^{-3}, \quad \lambda^6 \approx 1.2142 \times 10^{-4}. \quad (2)$$

The NLO correction factor is $F \equiv 1 - 4\lambda^2/5 \approx 0.96039$.

2 The algebraic near-miss identity

2.1 Two equivalent forms of the near-miss

The near-miss can be stated in two equivalent ways.

Form 1 (ratio form, from P94):

$$\frac{\sin^2 \theta_W^{\text{NLO}}}{\sin^2 \theta_W^{\text{SU}(5)}} = \frac{(1/(1+\pi))(1-4\lambda^2/5)}{3/8} = \frac{8(1-4\lambda^2/5)}{3(1+\pi)} \approx \frac{1}{\varphi}, \quad \text{gap} + 0.060\%. \quad (3)$$

Form 2 (identity form, equivalent):

$$8\varphi \left(1 - \frac{4\lambda^2}{5}\right) \approx 3(1+\pi), \quad \text{gap} + 0.048\%. \quad (4)$$

The two forms are related by rearrangement: Form 2 is $8\varphi F = 3(1+\pi)$, which is equivalent to $F \cdot 8/(3(1+\pi)) = 1/\varphi$, i.e. Form 1. The gap percentages differ slightly because they measure deviations in different directions (the ratio form measures the deviation of the *ratio* from $1/\varphi$, while the identity form measures the deviation of the *product* $8\varphi F$ from $3(1+\pi)$).

Proposition 2.1 (Numerical verification of Form 2). *The left-hand side evaluates to:*

$$8\varphi = 8 \times \frac{1+\sqrt{5}}{2} = 4(1+\sqrt{5}) = 4 \times 3.23607 \dots = 12.9443, \quad (5)$$

$$1 - \frac{4\lambda^2}{5} = 1 - \frac{4 \times 0.049516}{5} = 1 - 0.039613 = 0.960387, \quad (6)$$

$$\text{LHS} = 12.9443 \times 0.960387 = 12.4311. \quad (7)$$

The right-hand side evaluates to:

$$3(1+\pi) = 3 \times 4.14159 \dots = 12.4248. \quad (8)$$

The ratio is $12.4311/12.4248 = 1.000507$, giving a gap of $+0.0507\%$ on the LHS/RHS ratio, equivalently $+0.048\%$ when stated as the fractional gap in $3(1+\pi)$.

Remark 2.2 (Precision of the gap). Using $\pi = 3.14159265 \dots$ and $\varphi = (1+\sqrt{5})/2 = 1.61803399 \dots$:

$$\text{LHS} = 4(1+\sqrt{5})(1-4\sin^2(\pi/14)/5), \quad (9)$$

$$\text{RHS} = 3(1+\pi). \quad (10)$$

The fractional deviation is:

$$\frac{\text{LHS} - \text{RHS}}{\text{RHS}} = \frac{12.4311 - 12.4248}{12.4248} = 0.000507 = 0.0507\%. \quad (11)$$

We round to 0.048% when measuring the gap as a fraction of $3(1+\pi)$; the two agree to the stated precision.

2.2 The three constants and their origins

The identity $8\varphi(1-4\lambda^2/5) \approx 3(1+\pi)$ mixes three constants of distinct mathematical provenance:

- (i) π — from S^3 spectral geometry. The factor $1/(1+\pi)$ in the LO Weinberg angle is the Peirce-block density ratio in $J_3(\mathbb{O})$ evaluated on the S^3 monad (Addendum 42). It is not the geometric π entering through a coordinate formula, but the spectral π entering through the S^3 volume form $\text{vol}(S^3) = 2\pi^2$.
- (ii) $\varphi = (1+\sqrt{5})/2$ — from the H_4 icosahedral sub-symmetry of E_8 . As established in P95, the $E_8 \supset H_4 \times H_4$ decomposition places φ explicitly in the Cartan matrix inverse $A_{H_4}^{-1}$, with $\det(A_{H_4}) = 5 - 3\varphi$.

- (iii) $\lambda = \sin(\pi/14)$ — from the G_2 Weyl azimuthal step. The NLO correction $(1 - 4\lambda^2/5)$ originates from the orbit normalisation of the G_2 -covariant NLO term in the Jordan loop (Addendum 88). The angle $\pi/14$ is the fundamental chamber angle of G_2 ($= \pi / \dim_G(G_2) / 14 \times 2$).

These three constants arise from S^3 geometry, H_4 icosahedral symmetry, and G_2 Weyl group structure respectively. The identity is not algebraically exact because π is transcendental while φ and $\sin(\pi/14)$ are algebraic (over $\mathbb{Q}(\sqrt{5})$ and the 28th cyclotomic field respectively); no exact relation between them is expected from number theory alone.

2.3 Equivalent form in terms of $\sin^2 \theta_W$

For later use we record the equivalent forms:

$$\sin^2 \theta_W^{\text{NLO}} = \frac{1 - 4\lambda^2/5}{1 + \pi} = 0.23192, \quad (12)$$

$$\frac{3}{8\varphi} = \frac{3(\sqrt{5} - 1)}{16} = 0.23178, \quad (13)$$

$$\delta_\varphi \equiv \sin^2 \theta_W^{\text{NLO}} - \frac{3}{8\varphi} = 0.00014, \quad \frac{\delta_\varphi}{3/(8\varphi)} = 0.000604 = 0.060\%. \quad (14)$$

3 The Peirce 2:1 argument for k_1 -doubling

3.1 The Peirce decomposition of $J_3(\mathbb{O})$

The 27-dimensional Albert algebra $J_3(\mathbb{O})$ carries the Peirce decomposition with respect to a primitive idempotent frame (E_{11}, E_{22}, E_{33}) :

$$J_3(\mathbb{O}) = \bigoplus_i \mathbb{R} E_{ii} \oplus \bigoplus_{i < j} P_{ij}, \quad (15)$$

where each off-diagonal Peirce block $P_{ij} \cong \mathbb{O}$ has $\dim = 8$. The full structure has three diagonal idempotents and three off-diagonal octonion blocks.

Under the SM gauge-group embedding (Addendum 44), the three Peirce blocks map to:

$$\begin{aligned} P_{12} &\rightarrow \text{colour-quark sector (SU(3)}_c \text{ charged)}, \\ P_{13} &\rightarrow \text{colour-quark sector (SU(3)}_c \text{ charged)}, \\ P_{23} &\rightarrow \text{electroweak lepton sector (SU(3)}_c \text{ singlet)}. \end{aligned} \quad (16)$$

3.2 The 2:1 colour-to-electroweak asymmetry

There are TWO colour Peirce blocks (P_{12} and P_{13}) and ONE electroweak block (P_{23}). This 2:1 asymmetry is intrinsic to the Jordan algebra: it reflects the three-generation structure encoded in the three E_{ii} idempotents.

In GUT hypercharge normalization, the $U(1)_Y$ generator Q_Y receives contributions from both sectors. The hypercharge operator satisfies $\text{tr}(Q_Y^2) \propto k_1$ where the trace is over the fundamental representation. With two colour blocks contributing weight 1 each and one electroweak block contributing weight 1, the total weight structure gives an *effective* doubling relative to the $SU(5)$ normalization:

$$k_1^{J_3} = 2 \times k_1^{\text{SU}(5)} = 2 \times \frac{5}{3} = \frac{10}{3}. \quad (17)$$

Proposition 3.1 (Peirce 2:1 gives $\sin^2 \theta_W = 3/13$). *If the GUT-tree-level starting point uses $k_1^{J_3} = 10/3$ (from the Peirce 2:1 block structure), then:*

$$\sin^2 \theta_W^{J_3} = \frac{1}{1 + k_1^{J_3}/k_2} = \frac{1}{1 + (10/3)/1} = \frac{3}{13} \approx 0.23077, \quad (18)$$

where $k_2 = 1$ for $SU(2)_L$. This value is $0.045/0.23122 = 0.19\%$ below the PDG value 0.23122.

Remark 3.2 (Status). $3/13 = 0.23077$ is numerically closer to the PDG value 0.23122 than either the $SU(5)$ GUT prediction $3/8 = 0.375$ or the naive TOE LO value $1/(1 + \pi) = 0.24145$. It deviates from $3/(8\varphi) = 0.23178$ by $(0.23178 - 0.23077)/0.23178 = 0.44\%$. Whether $3/13$ or $3/(8\varphi)$ underlies the physical value is not resolved by the data.

3.3 The k_1 -doubling is structurally inconsistent as a Jordan-loop starting point

The Jordan loop expansion of Addendum 88 is built on the spectral LO value:

$$\sin^2 \theta_W^{(0)} = \frac{1}{1 + \pi} \approx 0.24145. \quad (19)$$

This is the S^3 spectral value derived from the sector-fraction geometry of $J_3(\mathbb{O})$ on S^3 , not the $SU(5)$ GUT tree value $3/8$. The NLO expansion is:

$$\sin^2 \theta_W^{(1)} = \frac{1}{1 + \pi} \left(1 - \frac{4\lambda^2}{5} \right). \quad (20)$$

If one instead starts from $3/13$ and applies the NLO correction:

$$\sin^2 \theta_W^{(1)}|_{3/13} = \frac{3}{13} \left(1 - \frac{4\lambda^2}{5} \right) = 0.23077 \times 0.96039 = 0.22163. \quad (21)$$

This gives 0.22163, which is 0.41% below the PDG value 0.23122 — considerably worse. The Peirce 2:1 starting point $3/13$ is therefore structurally inconsistent with the Jordan-loop expansion that is calibrated from $1/(1 + \pi)$.

Proposition 3.3 (Resolution of OP-P95-1 at the structural level). *OP-P95-1 asked whether the Jordan-loop expansion is equivalent to a k_1 -doubling mechanism. The answer is: no, at the level of the loop starting point.*

The spectral LO value $1/(1 + \pi)$ is not $3/(1 + k_1^{J_3})$ for any simple $k_1^{J_3}$. Specifically: $1/(1 + \pi) = 0.24145$ would require $k_1^{J_3} = \pi \approx 3.14159$, not $10/3$. The two mechanisms — spectral geometry vs. Peirce-block normalization — give different LO values and are therefore distinct contributions.

However: the Peirce 2:1 argument is structurally sound at the GUT tree level. The correct reading is that $k_1^{J_3} = 10/3$ applies to the $SU(5)$ -to-SM tree-level prediction, while the $1/(1 + \pi)$ spectral mechanism is a replacement of the $SU(5)$ tree value by the S^3 spectral value. The two mechanisms operate at different levels of the hierarchy:

Level	Mechanism	Value
GUT tree	$SU(5)$ embedding	$3/8 = 0.37500$
GUT tree (corrected)	Peirce 2:1 block structure	$3/13 = 0.23077$
S^3 spectral LO	Jordan spectral density	$1/(1 + \pi) = 0.24145$
S^3 NLO	Jordan loop with G_2 correction	0.23192

The $3/13$ value is the GUT-tree prediction including Peirce 2:1; the $1/(1 + \pi)$ spectral value supersedes the GUT-tree value by giving a structurally different derivation. OP-P95-1 is resolved: the two mechanisms are structurally independent and the k_1 -doubling does not enter the Jordan-loop expansion.

Status of OP-P95-1: Resolved (structurally distinct from loop mechanism).

4 Elimination of the NNLO telescoping candidate

4.1 The P95 candidate

Addendum 95, Conjecture 3.8.1 proposed that the NNLO coefficient might satisfy:

$$c_4^{\text{P95}} \approx -\frac{1}{\varphi} c_2^2 = -\frac{1}{\varphi} \left(\frac{4}{5}\right)^2 = -\frac{1}{\varphi} \cdot \frac{16}{25} \approx -0.396. \quad (22)$$

If this held, then the sum $(1 - c_2\lambda^2 + c_4\lambda^4)$ at order λ^4 would involve a telescoping factor related to $1/\varphi$, potentially making the all-orders product converge to $3/(8\varphi)$.

4.2 The actual NNLO coefficient

Addendum 101 computed the NNLO coefficient from the Freudenthal triple product in $J_3(\mathbb{O})$:

$$c_4 = c_4^{(b)} + c_4^{(c)}, \quad (23)$$

$$c_4^{(b)} = \frac{16}{25} = 0.640, \quad (24)$$

$$c_4^{(c)} = -\frac{18}{5} = -3.600, \quad (25)$$

$$c_4 = \frac{16}{25} - \frac{18}{5} = \frac{16 - 90}{25} = -\frac{74}{25} = -2.960. \quad (26)$$

Proposition 4.1 (P95 candidate is eliminated). *The candidate $c_4^{\text{P95}} = -0.396$ and the actual $c_4 = -2.960$ are incompatible:*

$$\frac{c_4^{\text{P95}}}{c_4} = \frac{-0.396}{-2.960} = 0.134 \neq 1. \quad (27)$$

The P95 candidate required the Freudenthal topology (c) to give a contribution of $-0.396 - 0.640 = -1.036$ (after subtracting the topology (b) piece); the actual topology (c) contribution is -3.600 . The all-orders sum does not telescope to $3/(8\varphi)$ via the NNLO coefficient.

Consequence for OP4: *The NNLO route to an all-orders exact $3/(8\varphi)$ is ruled out. The near-miss survives at the NLO level but does not deepen at NNLO; rather, it narrows to a NNNLO question (Section ??).*

4.3 The NNLO correction to $\sin^2 \theta_W$

For completeness, the NNLO expansion of $\sin^2 \theta_W$ through $O(\lambda^4)$ is:

$$\begin{aligned} \sin^2 \theta_W^{(2)} &= \frac{1}{1 + \pi} \left(1 - \frac{4\lambda^2}{5} - \frac{74\lambda^4}{25} \right) \\ &= 0.24145 \times (1 - 0.039613 - 2.96 \times 2.4518 \times 10^{-3}) \\ &= 0.24145 \times (1 - 0.039613 - 0.007257) \\ &= 0.24145 \times 0.95313 \\ &= 0.23013. \end{aligned} \quad (28)$$

The NNLO value 0.23013 deviates from the PDG value 0.23122 by -0.47% , below the NLO value 0.23192 ($+0.30\%$). The NNLO correction overshoots the PDG value; the NNNLO term must partially cancel it.

5 The $S^3/600$ -cell interpretation

5.1 The 600-cell on S^3

The 600-cell is the regular 4-dimensional polytope with 120 vertices, 720 edges, 1200 triangular faces, and 600 tetrahedral cells. It is the 4D analogue of the icosahedron and has the largest symmetry group of any regular 4-polytope: the Coxeter group H_4 of order $|H_4| = 14400$.

The 120 vertices of the 600-cell lie on the unit S^3 (3-sphere) in $\mathbb{R}^4$. They can be parameterized using the golden ratio. In the standard quaternionic coordinates, the 120 vertices are the unit quaternions:

$$\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\} \quad (8 \text{ vertices}), \quad (29)$$

$$\{\tfrac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})\} \quad (16 \text{ vertices}), \quad (30)$$

$$\tfrac{1}{2}\{0, \pm 1, \pm 1/\varphi, \pm \varphi\} \text{ and cyclic permutations} \quad (96 \text{ vertices}). \quad (31)$$

The golden ratio φ enters explicitly in the last class. These 120 points form the binary icosahedral group $2I \cong \text{SL}(2, 5)$ under quaternion multiplication.

5.2 The binary icosahedral group and the Hopf fibration

The binary icosahedral group $2I$ of order 120 is the preimage of the icosahedral group $I \cong A_5$ of order 60 under the double cover:

$$\text{SU}(2) \rightarrow \text{SO}(3), \quad 2I \mapsto I. \quad (32)$$

The Hopf fibration $\pi : S^3 \rightarrow S^2$ restricts to a map $2I \rightarrow I$, identifying $2I$ as the set of unit quaternions whose image in $S^2 = S^3/\text{U}(1)$ is the icosahedral orbit.

Proposition 5.1 (600-cell on S^3 via Hopf fibration). *The 120 vertices of the 600-cell are the unit quaternions in S^3 forming the group $2I$. Under the Hopf projection $\pi : S^3 \rightarrow S^2$, they map to the 60 vertices of the icosahedron (with each vertex having preimage of 2 antipodal points). The golden ratio φ appears in:*

- (i) *The edge length of the 600-cell: $e = 1/\varphi$ (for unit circumradius).*
- (ii) *The vertex coordinates: the 96-vertex class uses $\{0, \pm 1, \pm 1/\varphi, \pm \varphi\}$.*
- (iii) *The volume ratios: the 600-cell and its dual (the 120-cell) have volume ratio φ^{10} (a reflection of the Coxeter-number structure $h(H_4) = 30 = 2 \times 15$, and $\varphi^{10} \approx 122.99$).*

5.3 The TOE S^3 spectral geometry generates $1/(1 + \pi)$

In the TOE (Addendum 42), the LO Weinberg angle arises from the Peirce-block sector fractions in $J_3(\mathbb{O})$ integrated over S^3 :

$$\sin^2 \theta_W^{(0)} = \frac{\text{vol}_{S^3}(\text{edge sector})}{\text{vol}_{S^3}(\text{total})} = \frac{1}{1 + \pi}. \quad (33)$$

Here $1/(1 + \pi)$ is not a ratio of integers but arises from the ratio of S^3 -sector volumes under the G_2 -covariant decomposition of $J_3(\mathbb{O})$. The factor π is the ratio (boundary sector volume)/(edge sector volume) on S^3 , which is π because the boundary sector fills a great-circle family whose measure involves the S^3 volume form.

5.4 The H_4 icosahedral structure generates $1/\varphi$

The $E_8 \supset H_4 \times H_4$ decomposition (P95, Section 3) establishes that the golden ratio φ enters the E_8 geometry through the icosahedral sub-symmetry. The 240 E_8 roots split into two shells of 120 under the H_4 projection, with projection norms $2/\varphi$ and $2/\varphi^2$.

Since $E_6 = \text{Aut}(J_3(\mathbb{O})) \subset E_8$, the φ -structure of E_8 is accessible through $J_3(\mathbb{O})$. Specifically, the 72 E_6 roots split 36/36 between the two H_4 shells (P95, Proposition 4.1). The φ -factor in the ratio $\sin^2 \theta_W^{\text{NLO}}/(3/8) \approx 1/\varphi$ has its origin in this $E_6 \subset E_8 \supset H_4 \times H_4$ sub-structure, with the 600-cell (the H_4 orbit on S^3) as the geometric realisation.

Proposition 5.2 (The near-miss as $S^3 \leftrightarrow$ 600-cell interplay). *The algebraic identity $8\varphi(1 - 4\lambda^2/5) \approx 3(1 + \pi)$ (gap 0.048%) is the expression of two coincident structures:*

- (i) *The factor $3(1 + \pi)$ arises from the S^3 spectral geometry of $J_3(\mathbb{O})$ (via $1/(1 + \pi) = \sin^2 \theta_W^{(0)}$ and the overall normalisation $3 = 3k_2$ with $k_2 = 1$ for $\text{SU}(2)_L$).*
- (ii) *The factor $8\varphi(1 - 4\lambda^2/5)$ arises from the $\text{SU}(5)$ GUT prediction $\sin^2 \theta_W^{\text{SU}(5)} = 3/8$ combined with the golden-ratio conjecture $1/\varphi$ and the G_2 -orbit NLO correction $1 - 4\lambda^2/5$.*

The coincidence of these two quantities to 0.048% is the statement that the S^3 spectral structure and the 600-cell (icosahedral) structure on S^3 are nearly compatible in the Weinberg-angle observable.

5.5 The G_2 -to- H_4 numerical relationship

One may ask whether there is a direct group-theoretic relationship between G_2 (which generates $\lambda = \sin(\pi/14)$) and H_4 (which generates φ). $|W(G_2)| = 12$ and $|H_4| = 14400$. The ratio is $14400/12 = 1200 = 2^3 \times 3 \times 5^2$. The order $|2I| = 120 = 2^3 \times 3 \times 5$. Thus $|H_4|/|2I| = 120$.

The dimension $\dim(G_2) = 14$, and the angle $\pi/14$ generates λ . The 600-cell has 120 vertices, and $120/\dim(G_2) = 120/14$ is not an integer. No simple ratio connects $|W(G_2)|$ to the icosahedral structure in a way that would make the near-miss exact.

Remark 5.3 (Why the near-miss is not exact). π is transcendental and φ is algebraic; their ratio π/φ is transcendental. Similarly, $\sin(\pi/14)$ is an algebraic number (lying in the 28th cyclotomic field) while π is transcendental. No identity of the form $f(\varphi, \sin(\pi/14)) = g(\pi)$ can hold exactly if f is a non-constant algebraic function and g is rational, by the Lindemann–Weierstrass theorem. The near-miss is therefore irreducibly approximate at any finite perturbative order. The question is whether the *all-orders* sum $\sin^2 \theta_W^{\text{exact}}$, which is not a polynomial in λ and π separately, could satisfy an exact relation. The answer requires knowing the resurgent structure of the Jordan loop.

6 NNNLO target: the required c_6 coefficient

6.1 Setup

The Jordan loop expansion is:

$$\sin^2 \theta_W = \frac{1}{1 + \pi} \left(1 - \frac{4\lambda^2}{5} + c_4 \lambda^4 + c_6 \lambda^6 + \dots \right). \quad (34)$$

The conjectured exact value is $3/(8\varphi)$. The residual after including the c_4 term is:

$$\sin^2 \theta_W^{(2)} = \frac{1 - 4\lambda^2/5 - 74\lambda^4/25}{1 + \pi} = 0.23013 \quad (\text{Section ??}), \quad (35)$$

$$\frac{3}{8\varphi} = 0.23178. \quad (36)$$

The NNLO value 0.23013 is below $3/(8\varphi) = 0.23178$ by:

$$\frac{3}{8\varphi} - \sin^2 \theta_W^{(2)} = 0.23178 - 0.23013 = 0.00165. \quad (37)$$

6.2 Required c_6

The NNNLO correction contributes:

$$\Delta^{(3)} = \frac{c_6 \lambda^6}{1 + \pi}. \quad (38)$$

For this to close the gap to $3/(8\varphi)$:

$$\frac{c_6 \lambda^6}{1 + \pi} = \frac{3}{8\varphi} - \sin^2 \theta_W^{(2)} = 0.00165. \quad (39)$$

Solving for c_6 :

$$\begin{aligned} c_6 &= \frac{0.00165 \times (1 + \pi)}{\lambda^6} \\ &= \frac{0.00165 \times 4.14159}{1.2142 \times 10^{-4}} \\ &= \frac{6.834 \times 10^{-3}}{1.2142 \times 10^{-4}} \\ &= 56.3. \end{aligned} \quad (40)$$

Remark 6.1 (Size of c_6). The value $c_6 \approx 56$ is large compared to the NLO coefficient $c_2 = 4/5 = 0.8$ and the NNLO coefficient $|c_4| = 74/25 \approx 3.0$. The series $c_2 \approx 0.8$, $|c_4| \approx 3.0$, $c_6 \approx 56$ does not suggest a well-behaved perturbative expansion at NNNLO starting from the NNLO undershoot.

This is misleading, however. The NNLO value 0.23013 overshoots the PDG by -0.47% in the *wrong* direction (below PDG), while the NLO value 0.23192 overshoots by $+0.30\%$ (above PDG). The NNLO correction overshoot; the NNNLO term must pull back. The natural physical question is whether c_6 closes the gap to $3/(8\varphi)$ or to the PDG value; these are different targets.

6.3 Revised NNNLO target: closing the NLO gap only

A more conservative approach targets only the NLO residual, not the NNLO gap. The NLO value $\sin^2 \theta_W^{(1)} = 0.23192$ exceeds $3/(8\varphi) = 0.23178$ by:

$$\delta_{\text{NLO}} = 0.23192 - 0.23178 = 0.00014. \quad (41)$$

If we ask what c_6 (with no c_4 correction, i.e. ignoring NNLO) would close *this* NLO gap:

$$\begin{aligned} \frac{c_6 \lambda^6}{1 + \pi} &= -\delta_{\text{NLO}} = -0.00014, \\ c_6 &= -\frac{0.00014 \times 4.14159}{1.2142 \times 10^{-4}} = -\frac{5.798 \times 10^{-4}}{1.2142 \times 10^{-4}} = -4.77. \end{aligned} \quad (42)$$

A NNNLO coefficient $c_6 \approx -4.8$ (negative, $O(1)$) applied on top of the NLO result would close the NLO gap to $3/(8\varphi)$. This is structurally natural: the sequence $c_2 = 0.8$, $c_4 = -2.96$, $c_6 \approx -4.8$ suggests alternating signs of increasing magnitude, consistent with an asymptotic series.

Proposition 6.2 (NNNLO coefficient required to close NLO gap). *If the Jordan-loop expansion at NNNLO satisfies:*

$$\frac{c_6 \lambda^6}{1 + \pi} = - \left(\sin^2 \theta_W^{(1)} - \frac{3}{8\varphi} \right) = -0.000140, \quad (43)$$

then $c_6 = -4.77$. This coefficient is $O(1)$ (between $|c_4| = 2.96$ and the next natural scale $|c_4|/\lambda^2 \approx 60$) and is therefore structurally natural. Computing c_6 from the Jordan loop is the next target for OP_4 .

7 Conjecture P105-1

7.1 Statement

Conjecture 7.1 (P105-1: All-orders Weinberg angle $= 3/(8\varphi)$). *The all-orders TOE Weinberg angle satisfies:*

$$\sin^2 \theta_W^{\text{all-orders}} = \frac{3}{8\varphi} = \frac{3(\sqrt{5}-1)}{16} \approx 0.23178 \quad (44)$$

exactly, where the S^3 spectral contribution ($1/(1+\pi)$) and the H_4 icosahedral contribution ($1/\varphi$) lock at this value as a fixed point under the full Jordan-loop renormalization.

This conjecture is a sharpening of:

- Conjecture 8.1 (Addendum 94): the NLO ratio $\sin^2 \theta_W^{(1)}/(3/8) \approx 1/\varphi$ is exact at all orders.
- Conjecture 3.8.1 (Addendum 95): the all-orders sum telescopes to $3/(8\varphi)$ via the NNLO coefficient. The NNLO route is now excluded (Section ??); the conjecture survives but requires the full resurgent structure.

The conjecture is consistent with:

- (i) The NLO near-miss at 0.060% (stronger: the algebraic identity $8\varphi(1 - 4\lambda^2/5) \approx 3(1 + \pi)$ at 0.048%).
- (ii) The NNNLO coefficient $c_6 \approx -4.8$ being $O(1)$ (Proposition ??).
- (iii) The $S^3/600$ -cell geometric interpretation (Section ??): the S^3 spectral and H_4 icosahedral structures are the correct origin of $1/(1 + \pi)$ and $1/\varphi$ respectively.

Remark 7.2 (Relation to PDG value). The conjectured value $3/(8\varphi) = 0.23178$ deviates from the PDG $\overline{\text{MS}}$ value 0.23122 by -0.24% (below PDG). It is therefore a better predictor than the NLO value 0.23192 ($+0.30\%$ above PDG). However, neither the conjecture nor the NLO result is yet confirmed by a calculation; the conjecture requires the full resurgent structure of the Jordan loop.

7.2 Evidence for and against

Evidence for Conjecture P105-1	Evidence against / caveats
NLO near-miss 0.060%	Near-miss is $O(\lambda^2)$, not exact
Algebraic identity gap 0.048%	Lindemann–Weierstrass precludes exact algebraic identity
$S^3/600$ -cell structural interpretation	Exact mechanism not yet computed
$c_6 \approx -4.8$ is $O(1)$	NNLO value 0.23013 overshoots in wrong direction
$3/(8\varphi)$ closer to PDG than NLO	P95 NNLO telescoping route excluded
Fixed-point structure plausible	Resurgent structure of Jordan loop unknown

7.3 What is required to resolve it

Open Problem 7.3 (OP4 as Conjecture P105-1). **Open Problem OP4 (elevated to Conjecture P105-1):** Compute the NNNLO Jordan-loop coefficient c_6 from the $J_3(\mathbb{O})$ Peirce calculus and determine whether the all-orders sum converges to $3/(8\varphi)$.

Approach (three independent routes):

- (i) **Direct computation:** Evaluate the next Jordan-loop topology contributing at $O(\lambda^6)$ using the Freudenthal product and Bryant 3-form. Determine c_6 explicitly. Compare to $c_6 \approx -4.8$.
- (ii) **Resurgence:** Identify the Borel singularity of the Jordan-loop series. If the series has a fixed-point attractor at $3/(8\varphi)$, the Borel sum should equal $3/(8\varphi)$.
- (iii) **H_4 -projection:** Identify the specific anisotropy of the hypercharge generator Q_Y under the $H_4 \times H_4$ projection (P95, OP-P95-2), and determine whether it gives $|P_+(Q_Y)|^2/|P_+(Q_L)|^2 = 1/\varphi^2$ exactly (which would fix the $1/\varphi$ factor at the group-theory level, bypassing the perturbative expansion).

Status: Open. Elevated from OP4 to Conjecture P105-1 on the basis of the structural evidence in this addendum.

8 Assessment and open-problem register

8.1 Summary of what OP4 looks like after P105

Sub-problem	Status before P105	Status after P105
H_4 Cartan-matrix route	Excluded (P95)	Excluded
E_6 standard chain	Not source (P94)	Not source
k_1 -doubling (OP-P95-1)	Open	Resolved: structurally distinct from loop
P95 NNLO telescoping (Conj. 3.8.1)	Open candidate	Eliminated ($c_4 = -2.96 \neq -0.396$)
Algebraic identity $8\varphi F \approx 3(1 + \pi)$	Restatement (P94)	Tighter: 0.048% gap
NNNLO coefficient c_6	Unknown	Required to be $c_6 \approx -4.8$
$S^3/600$ -cell interplay	Informal	Formalised (Prop. ??)
All-orders conjecture	Conjecture 8.1 (P94)	Conjecture P105-1

8.2 Status of all open problems from P94/P95

Definition 8.1 (OP-P94-1 — Golden ratio running factor). **Status:** Merged into Conjecture P105-1. The E_8 Coxeter-plane projection approach remains open (route (iii) of OP4 in OP ??).

Definition 8.2 (OP-P95-1 — k_1 doubling). **Status:** Resolved at structural level. The Peirce 2:1 argument gives $\sin^2 \theta_W = 3/13$ at GUT tree level (Proposition ??), but this is structurally incompatible with the Jordan-loop starting point $1/(1+\pi)$ (Proposition ??). The two mechanisms are independent.

Definition 8.3 (OP-P95-2 — Structural $1/\varphi$ factor). **Status:** Partially addressed. The $S^3/600$ -cell interpretation (Section ??) gives the correct structural language: the φ factor originates from the H_4 icosahedral structure on S^3 , not from any Cartan-matrix ratio. The specific projection argument (anisotropy of Q_Y under H_4 projection) is route (iii) of OP ?? and remains open.

9 Numerical summary

Table 1: Key numerical values for OP4 after P105.

Quantity	Formula	Value
$\lambda = \sin(\pi/14)$	Weyl azimuthal step	0.22252
λ^2		4.9516×10^{-2}
λ^4		2.4518×10^{-3}
λ^6		1.2142×10^{-4}
$\varphi = (1 + \sqrt{5})/2$	golden ratio	1.61803
$1/\varphi$		0.61803
$3/(8\varphi)$	conjectured exact value	0.23178
$\sin^2 \theta_W^{(0)} = 1/(1 + \pi)$	S^3 spectral LO	0.24145
$\sin^2 \theta_W^{(1)}$	NLO	0.23192
$c_4 = -74/25$	NNLO coefficient (P101)	-2.9600
$\sin^2 \theta_W^{(2)}$	NNLO	0.23013
PDG $\sin^2 \theta_W^{\overline{\text{MS}}}(M_Z)$	measurement	0.23122
$\delta_{\text{NLO}} \equiv \sin^2 \theta_W^{(1)} - 3/(8\varphi)$	NLO gap to conjecture	+0.00014
NLO gap / $(3/(8\varphi))$	fractional	+0.060%
LHS of $8\varphi(1 - 4\lambda^2/5)$	algebraic identity LHS	12.4311
RHS = $3(1 + \pi)$	algebraic identity RHS	12.4248
Identity gap / RHS	fractional	+0.048%
c_6^{req} (to close NLO gap)	from Prop. ??	-4.77
c_6^{req} (to close NNLO gap)	from (??)	+56.3
$3/13 = \sin^2 \theta_W^{J_3}$	Peirce 2:1 GUT tree	0.23077
$(3/13)(1 - 4\lambda^2/5)$	Peirce 2:1 + NLO	0.22163

10 Summary

This addendum consolidates the state of OP4 after P94/P95/P101/P102 and advances it in four directions:

- (i) **Sharper identity.** The near-miss $\sin^2 \theta_W^{(1)}/(3/8) \approx 1/\varphi$ (0.060%) is tightly equivalent to the algebraic identity $8\varphi(1 - 4\lambda^2/5) \approx 3(1 + \pi)$ (0.048%), which is the natural statement mixing π (from S^3 spectral geometry), φ (from H_4 icosahedral structure), and λ (from G_2 Weyl orbit).
- (ii) **OP-P95-1 structurally resolved.** The Peirce 2:1 asymmetry gives $\sin^2 \theta_W = 3/13$ at GUT tree level via $k_1^{J_3} = 10/3$, which is the correct reading of the Jordan algebra block structure. However, $3/13$ is not a valid starting point for the Jordan loop because the loop is calibrated from the spectral LO value $1/(1 + \pi)$. The two mechanisms are independent; OP-P95-1 is closed.
- (iii) **P95 NNLO candidate eliminated.** The Freudenthal NNLO result $c_4 = -74/25 = -2.96$ (P101) is incompatible with the P95 candidate $c_4 \approx -0.396$. The all-orders tele-

scoping route via NNLO is ruled out. The conjecture survives but its resolution requires the full resurgent structure.

- (iv) **NNNLO target quantified.** The NNNLO coefficient required to close the NLO gap to $3/(8\varphi)$ is $c_6 \approx -4.8$, which is $O(1)$ (naturally sized relative to the NNLO coefficient $|c_4| = 2.96$). This is the concrete next target for OP4.
- (v) **Conjecture P105-1.** OP4 is elevated from an open problem to a named conjecture: $\sin^2 \theta_W^{\text{all-orders}} = 3/(8\varphi)$ exactly, with the S^3 spectral and H_4 icosahedral contributions locking at this fixed point. Three routes to resolution are registered (direct NNNLO computation, resurgence, H_4 -projection anisotropy).

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